For calculating Nontrivial Zeros of Riemann Zeta function-ζ, the definition $\xi(s)=\frac{s}{2}(s-1)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s)$ of Riemann Xi function-ξ is not appropriate.

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ABSTRACT

We show that for calculating nontrivial zeros of the Riemann Zeta function ζ, the form of the definition $\xi(s)=(s/2)(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s)$, $s\in\mathbb{C}$ of the function ξ and the followed deduction that nontrivial zeros of functions $\zeta(s)$ and $\xi(s)$ are identical is not appropriate. The definition of function ξ in which both functions ξ and ζ are functions of same complex variable s and the assumption of identicalness of nontrivial zeros of ξ and ζ is ambiguous, so may be the deep reason, the Riemann hypothesis could not be resolved yet. However, the definition $\xi(t)=(s/2)(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s)$, $t=\alpha+i\beta$ and $s=\underline{1/2}+it$ introduced by B. Riemann (1859) leads the results: (i) when $\beta=0$ and $\xi(\alpha)=0$, corresponding nontrivial zero of function $\zeta(s)$ are of the form $s=\underline{1/2}+i\alpha$ and (ii) when $t=\alpha+i\beta$ and $\xi\ \alpha+i\beta\ =0$, nontrivial zeros of the function $\zeta(s)$ are of the form $s=(\underline{1/2}-\beta)+i\alpha$ which lie on both sides of the line $\alpha=1/2$. Here, we sketch the zeros of the function $\zeta(s)$ those correspond to real zeros of the function $\xi(s)$ that shows the Riemann hypothesis is true only when nontrivial zeros of functions $\xi(s)$ and $\zeta(s)$ lie on the lines perpendicular to each other.

Keywords: Zeta function, Riemann's Xi Function, nontrivial zeros, critical strip, critical line.

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1 INTRODUCTION

In 1859, B. Riemann [1] in his research paper introduced a function $\zeta\ s\ \ s=\sigma+it,\sigma,t\in\mathbb{R}$ known as the Riemann's zeta function $\zeta\ s$ with the definition,

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s} \quad \ldots (1)$$

Riemann further created another function known as the Xi-function $\xi\ t\ ,t=\alpha+i\beta$ defined as:

$$\xi\ t\ =\ s/2\ \ s\text{-}1\ \pi^{\text{-}s/2}\zeta\ s\ ,s=\underline{1/2}+it \quad \ldots (2)$$

The definition (2) is the **original definition** of the function ξ. But in Mathematics literature present day authors e.g. [2], [3], [4] and other use an alternative following definition,

$$\xi\ s\ =\ s/2\ \ s\text{-}1\ \pi^{\text{-}s/2}\zeta\ s \quad ... (3)$$

With the definition (3) authors claim that nontrivial zeros of the functions $\xi\ s$ and $\zeta\ s$ are identical.

In this research article, we show that the use of the definition (3) of the function-ξ cannot be justified as it creates some mathematical ambiguities. However, the original definition (2) of the function ξ and corroborated with Riemann's statement: " it is clear that $\xi\ t$ can vanish only if the imaginary part of t lies between $i/2$ and $-i/2$." which indicates that t is a complex variable, produces the results : (i) Corresponding to each complex zero $t=\alpha+i\beta$ of the function $\xi\ t$, there exists a complex zero $s=\ 1/2\text{-}\beta\ +i\alpha$ of the function $\zeta\ s$, i.e., zeros of functions $\xi\ t$ and $\zeta\ s$ are at a distance apart. (ii) Corresponding to each real zero $t=\alpha,\alpha\in\mathbb{R}$ of the function $\xi\ t$, there exists a complex zero $s=\underline{1/2}+it$ of function $\zeta\ s$.

Trial zeros obtained in results (i) and (ii), are sketched in Fig. 1(a) and 1(b).

2 RESULTS

Recall the definition (3) connecting functions ξ and ζ both of same complex variable s,

$$\xi(s) = (s/2)(s-1)\pi^{-s/2}\zeta(s),\ s=\mu+i\lambda \qquad \ldots (4)$$

Clearly, $\xi(0)=0$ and $\xi(1)=0$, i.e., $s=0, s=1$ are real zeros of $\xi(s)$. Suppose zeros functions $\xi(s)$ and $\zeta(s)$ are identical then $s=0, s=1$ must also be zeros of $\zeta(s)$ but according to definition (1) of $\zeta(s)$, $\zeta(0)=\infty$ and $\zeta(1)=\infty$, therefore $s=0, s=1$ are not zeros of $\zeta(s)$, so not of the function $\xi(s)$. That is ambiguity in definition (4). Actually, when s is a real number, all zeros of $\zeta(s)$ necessarily are zeros of the function $\xi(s)$ but when s is a complex number zeros of functions $\xi(s)$ and $\zeta(s)$ may be different that is shown here:

Suppose $\xi = G+iH$, $(s/2)(s-1)\pi^{-s/2} = C+iD$ and $\zeta(s) = A+iB$, then from result (4),

$$G+iH = (CA-DB)+i(AD+BC) \qquad \ldots (5)$$

Zeros of $\xi(s)$ can be obtained choosing G=0 and H=0 which means $CA-DB=0$ and $AD+BC=0$. This system of equations produces $A=0$, $B=0$, $A=iB$, $C=iD$, $C=0$, and $D=0$. Moreover, the function $\zeta(s)$ can be written as $\zeta(s)=\sqrt{A^2+B^2}(\cos\phi+i\sin\phi)$ with $\phi=\tan^{-1}\left(\frac{B}{A}\right)$. Now, if $\xi(s)=0$, $(s/2)(s-1)\pi^{-s/2}\neq 0$, then $\sqrt{A^2+B^2}(\cos\phi+i\sin\phi)=0$ which implies the equation $\zeta(s)=0$ is unsolvable.

Further, suppose that $s=a_i, a_i\in\mathbb{R}\text{ or }\mathbb{C}$, $i=1,2,3,\ldots,n$ are zeros of the function $\xi(s)$ and $s=b_j, b_j\in\mathbb{R}\text{ or }\mathbb{C}$, $j=1,2,3,\ldots,m$ are zeros of the function $\zeta(s)$, i.e., $\xi(s)=\prod_{i=1}^{n}(s-a_i)$ and $\zeta(s)=\prod_{j=1}^{m}(s-b_j)$. Therefore, the result (4) can be expressed as,

$$\prod_{i=1}^{n}(s-a_i) = (s/2)(s-1)\pi^{-s/2}\Gamma(s/2)\prod_{j=1}^{m}(s-b_j) \qquad \ldots (6)$$

There are two cases:

Case I: At least one zero $s = a$ (say) is common to both functions ξ and ζ then,

$\xi(s) = (s-a)\prod_{i=1}^{n-1}(s-a_i)$ and $\zeta(s) = (s-a)\prod_{i=1}^{m-1}(s-b_i)$, therefore,

$$(s-a)\prod_{i=1}^{n-1}(s-a_i) = (s/2)(s-1)\pi^{-s/2}\Gamma(s/2)(s-a)\prod_{i=1}^{m-1}(s-b_i) .$$

Further, write $\prod_{i=1}^{n-1}(s-a_i) = \xi_1(s)$ and $\prod_{i=1}^{m-1}(s-b_i) = \zeta_1(s)$, then

$$\left.\begin{array}{l} (s-a)\left[\xi_1(s) - (s/2)(s-1)\pi^{-s/2}\Gamma(s/2)\zeta_1(s)\right] = 0 \\ \left[\xi_1(s) - (s/2)(s-1)\pi^{-s/2}\Gamma(s/2)\zeta_1(s)\right]_{s=a} = \left(0/(s-a)\right)_{s=a} \\ \left[\xi_1(a) - (a/2)(a-1)\pi^{-\alpha/2}\Gamma(a/2)\zeta_1(a)\right] = 0/0 \end{array}\right\} \qquad \ldots (7)$$

Thus there exists at least one case that when $s = a$ the quantity

$\left[\xi_1(a) - (a/2)(a-1)\pi^{-\alpha/2}\Gamma(a/2)\zeta_1(a)\right]$ is not non-zero but indeterminate. However, in general the quantity $\left[\xi_1(a) - (a/2)(a-1)\pi^{-\alpha/2}\Gamma(a/2)\zeta_1(a)\right]$ is considered non-zero.

Case II: Functions $\xi(s)$ and $\zeta(s)$ have same number of identical zeros. Let γ be one of such zeros, then

$$\prod_{\gamma}(s-\gamma)\left[1-(1/2)s(s-1)\pi^{-s/2}\Gamma(s/2)\right]=0$$

$$1-(1/2)\gamma(\gamma-1)\pi^{-\gamma/2}\Gamma(\gamma) = 0/0 \qquad \ldots (8)$$

If $1-(1/2)\gamma(\gamma-1)\pi^{-\gamma/2}\Gamma(\gamma)$ is nonzero then from result (8), either $0=0$ or

$1-(1/2)\gamma(\gamma-1)\pi^{-\gamma/2}\Gamma(\gamma)$ is indeterminate. Also, if γ equals 1, then $1=0/0$ and if

$1-(1/2)\gamma(\gamma-1)\pi^{-\gamma/2}\Gamma(\gamma)$ equals zero then $0=0/0$, i.e. 0 is itself indeterminate.

Whatever be the case I or II discussed above but even one common zero $s=\alpha$ results $\left[\xi_1(\alpha)-(1/2)\alpha(\alpha-1)\pi^{-\alpha/2}\Gamma(\alpha/2)\zeta_1(\alpha)\right]=0/0$ which shows 0 is not a free number, its use is conditional. Thus, from the above discussion it can be concluded that (i) the definition (4) of the function ξ is not a proper definition for calculating nontrivial zeros of the function $\zeta(s)$ and (ii) to solve an equation like $f(x)\cdot g(x)=0$, $f(x)$ or $g(x)\in\mathbb{C}$, the definition of zero requires investigation because the conclusion from the equation $X+iY=0$, $X,Y\in\mathbb{C}$ implies $X=0$ and $Y=0$ is not always true. The consideration $\zeta(s)=X+iY=0$ implies $X=0$ and $Y=0$ for the function $\zeta(s)$ is the foremost reason; the Riemann hypothesis could not have been resolved yet, also the claimed nontrivial zeros 14.134725142, 21.022039639, 25.010857580, and so on are not nontrivial zeros of the function $\zeta(s)$ that we will show elsewhere.

Now, using the definition, $\xi(t)=(s/2)(s-1)\pi^{-s/2}\zeta(s)$ we establish a relation between nontrivial zeros of function $\xi(t)$ and $\zeta(s)$, $s=\underline{1/2}+it$.

Riemann states: “It is clear that $\xi(t)$ can vanish only if the imaginary part of t lies between $i/2$ and $-i/2$.” That suggests t is a complex variable. Suppose $t=\mu+i\lambda$ (say) and $\zeta(s)$, $s=\underline{1/2}+it$. Therefore, from the definition (3),

$$\xi\ \mu+i\lambda\ =\ 1/2\ \ \underline{1/2}-\lambda+\mu i\ \ \underline{1/2}-\lambda+\mu i-1\ \pi^{-\ \underline{1/2}-\lambda+\mu i\ /2}\Gamma\Big[\ 1/2\ \ \underline{1/2}-\lambda+\mu i\ \Big]\zeta\ \underline{1/2}-\lambda+\mu i$$

Substitute, 0 for μ and $1/2$ for λ (or $t = i/2$)

$$\xi\ i/2\ =\ 1/2\ \ 0\ \ -1+0i\ \pi^{0}\Gamma\ 0\ \zeta\ 0\ =0 \qquad \ldots (9)$$

Substitute, 0 for μ and $-1/2$ for λ (or $t = -i/2$)

$$\xi\ -i/2\ =\ 1/2\ \ 1\ \ 0\ \pi^{-1/2}\Gamma\Big[\ 1/2\ \ 1\ \Big]\zeta\ 1\ =0 \qquad \ldots (10)$$

That shows $t = -i/2$ and $t = i/2$ are nontrivial zeros of the function $\xi\ t$ but corresponding to $\xi\ -i/2$ and $\xi\ i/2$ the values $\zeta\ 0$ and are $\zeta\ 1$ undefined. To avoid this ambiguity Riemann states: "$\xi\ t$ can vanish only if the imaginary part of t lies between $i/2$ and $-i/2$".

The results (9) and (10) show if nontrivial zeros of function $\xi\ t$ lie between $t = -i/2$ to $t = i/2$, then corresponding zeros of the function $\zeta\ s$ lie between $s = 1$ to $s = 0$ .Thus, the range of nontrivial zeros of the function $\zeta\ s$ is $s \in\ 0,1$ which is the critical strip for nontrivial zeros of function $\zeta\ s$. The critical strip for nontrivial zeros of $\zeta\ s$ can also be determined as:

Suppose $t = \alpha \pm i\beta$ are zeros of the function $\xi\ t$, then according to Riemann's statement,

$$\begin{aligned}
&-i/2 \le t \le i/2 \\
&\qquad \Rightarrow i^2 1/2 \le -it \le -i^2 1/2 \\
&\qquad \Rightarrow -1/2 \le -i\ \ \alpha \pm i\beta\ \ \le 1/2 \\
&\qquad \Rightarrow -1/2 \le -i\alpha \mp \beta \le 1/2 \\
&\qquad \Rightarrow 1/2 \ge i\alpha \pm \beta \ge -1/2 \\
&\qquad \Rightarrow 1 \ge\ 1/2 \pm \beta\ + i\alpha \ge 0 \\
&\qquad \Rightarrow 0 \le\ 1/2 \pm \beta\ + i\alpha \le 1
\end{aligned}$$

But $1/2 \pm \beta + i\alpha$ is variable of the function $\zeta(s)$ corresponding to $t = \alpha \pm i\beta$. Therefore, if zeros of function $\xi(t)$ lie between $t = -i/2$ to $t = i/2$, then zeros of the function $\zeta(s)$ lie between $s = 0$ to $s = 1$.

Thus, nontrivial zeros of the function $\zeta(s)$ are of the form $\underline{1/2} \mp \beta + i\alpha$ that lie in the region $0 \le \underline{1/2} \mp \beta \le 1$ that verbalize the Riemann hypothesis. Further, if β equals zero, i.e. all zeros of function $\xi(t = \alpha \pm i\beta)$ are real then zeros of $\zeta(s = \underline{1/2} + it)$ are of the form $\underline{1/2} + i\alpha$ that lie in the region $0 \le 1/2 \le 1$ on the line $a = 1/2$. Clearly, the functions $\xi(t = \alpha \pm i\beta)$ and $\zeta(s = \underline{1/2} + it)$ have same number of zeros and there is one-to-one correspondence between real zeros of the function ξ and nontrivial complex zeros of the function $\zeta(s)$.

Nontrivial zeros of functions $\xi(t)$ and $\zeta(s)$ when (i) t is a complex number, and (ii), when t is real number are in Fig, 1(a) and Fig. 1(b) respectively. Here, for to show the relative locations of zeros of the function $\zeta(s)$, zeros of the function $\xi(t)$ are arbitrary.

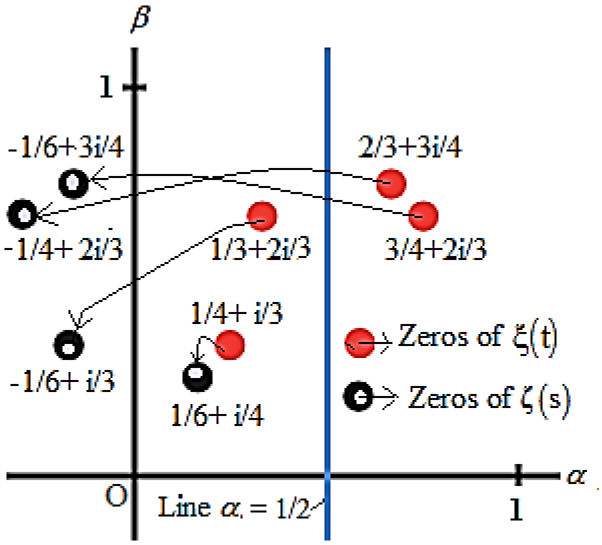

Fig. 1(a): Zeros of functions $\xi(t)$ and $\zeta(s)$ when t is a complex variable

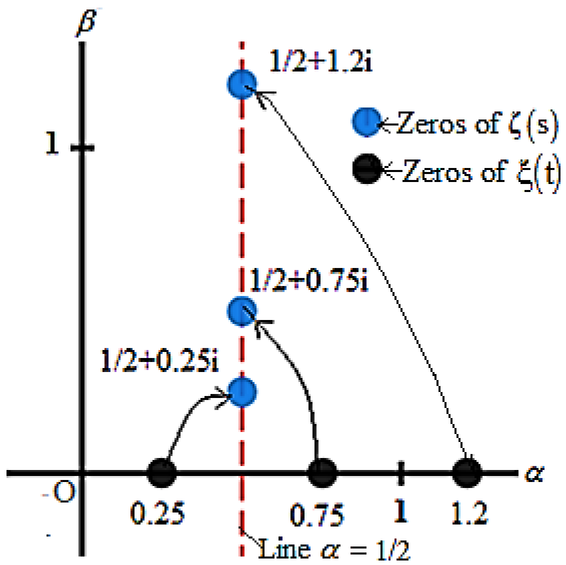

Fig. 1(b): Zeros of functions $\xi(t)$ and $\zeta(s)$ when t is a real variable

Thus, if $t = \alpha \pm i\beta$ is zero of the function $\xi \;\; t$, then corresponding zero of the function $\zeta \;\; s$

is $s = \left(\frac{1}{2} \mp \beta\right) \pm i\alpha$. That show zeros of functions ξ and ζ cannot have same form and same

variable and in the context of the Riemann hypothesis the form of definition of function ξ

$\xi \;\; s \;=\; s/2 \;\; s\text{-}1 \;\; \pi^{-s/2} \;\; \zeta \;\; s \;,\; s = \mu + i\lambda$ is ambiguous.

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Declaration

The Author does not have any compelling interest writing this research article. The Author communicates this research article through this pre-print repository to share the knowledge to the interested audience.

Additional Information:

Corresponding to non-trivial zero $\alpha + i\beta$ of the function ξ, non-trivial zero of the function ζ is $\left(\frac{1}{2} - \beta\right) + i\alpha$.

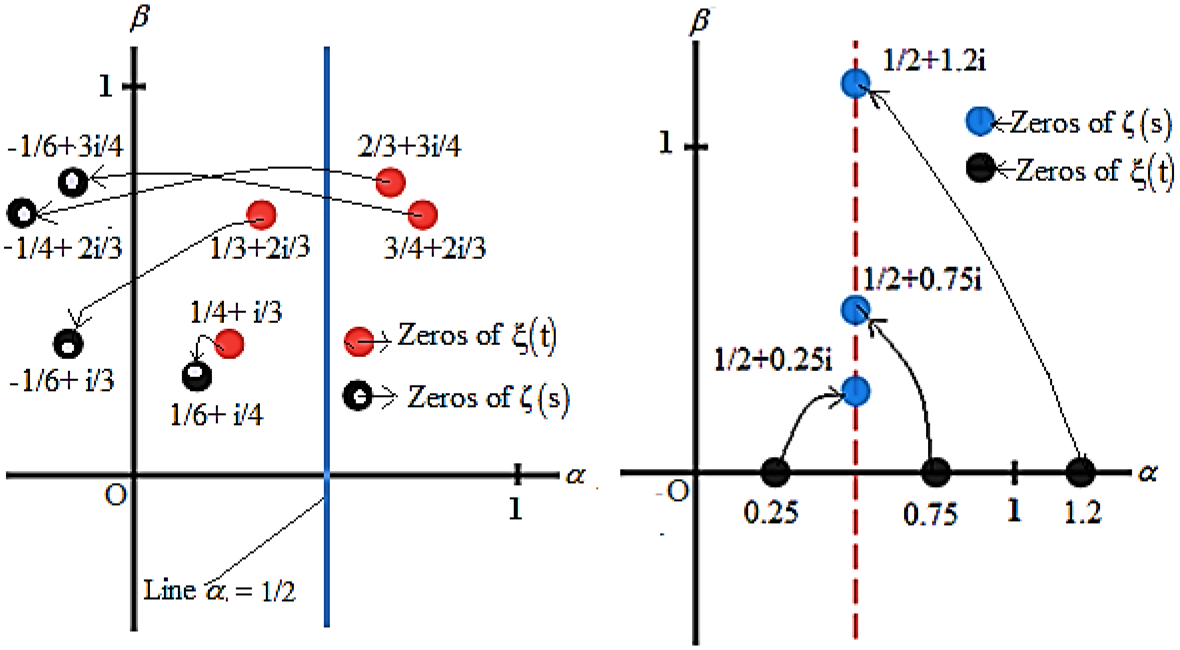

Fig. 1(a): Zeros of functions $\xi(t)$ and $\zeta(s)$ when t is a complex variable

Fig. 1(b): Zeros of functions $\xi(t)$ and $\zeta(s)$ when t is a real variable