Remarks on a formula of Ji-Cai Liu

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Abstract: Via symbolic summation method, we establish many series for π^2 .

1. Introduction

In [1] , (arXiv: 2002.03650v1 , 10 Feb 2020) , Ji-Cai Liu proved that

$$\sum_{n=1}^{\infty} \frac{H_n - 2 H_{2n}}{(-3)^n n} = \frac{\pi^2}{18} \tag{1}$$

where

$$\pi = 4 \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \tag{2}$$

$$H_n = \sum_{k=1}^{n} \frac{1}{k} \ , \ n = 1, 2, 3, \ \ldots; \ H_0 = 0 \tag{3}$$

In this note we give some formulas related to (1).

2. Formulas

Entry 1.

$$\frac{\pi^2}{36} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{3^n n} \sum_{k=1}^{n} \frac{1}{2k-1} \tag{4}$$

Entry 2.

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{3^n} (6 H_{2n} + 2 H_{2n+2} - 3 H_n - H_{n+1}) H_n \tag{5}$$

$$\frac{\pi^2}{12} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{3^n} \left(4 H_{2n} - 2 H_n + \frac{1}{2n+1}\right) H_n \tag{6}$$

Entry 3.

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{3^n} (6 H_{2n} - 2 H_{2n+2} - 3 H_n + H_{n+1}) h_n \tag{7}$$

$$\frac{\pi^2}{12} = \sum_{n=1}^{\infty} \frac{1}{3^n} \left(2 H_{2n} - H_n - \frac{1}{2n+1}\right) h_n \tag{8}$$

where

$$h_n = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} \ , \ n = 1, 2, 3, \ \ldots \tag{9}$$

Entry 4.

$$\frac{\pi^2}{9} = \sum_{n=1}^{\infty} \frac{3^{-2n}}{n(2n-1)} \left((4n+1)(2H_{4n-2} - H_{2n-1}) - \frac{4n-2}{4n-1} \right) \tag{10}$$

Entry 5.

$$\frac{\pi^2}{18} = \sum_{n=2}^{\infty} \frac{(-1)^n}{3^n} \sum_{k=1}^{[n/2]} \frac{(-1)^{k-1} 3^k}{n-k} \left(\frac{1}{k} + \frac{2}{n} \right) \tag{11}$$

$$\frac{\pi^2}{18} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{3^n n} \sum_{k=1}^{n} \frac{n+3k}{k(n+k)} \tag{12}$$

Entry 6.

$$\frac{\pi^2}{12} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(4n+3)}{3^n n(n+1)} \sum_{k=1}^{n} \frac{k}{2n-2k+1} \tag{13}$$

$$\frac{\pi^2}{12} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(2n+3)}{3^n n(n+1)} \sum_{k=1}^{\left[\frac{n+1}{2}\right]} \frac{1}{4k-2+(-1)^n} \tag{14}$$

Entry 7. for $1/3 < x < 1$, we have

$$\frac{\pi^2}{18} = (1-x) \sum_{n=1}^{\infty} x^n (2H_{2n} - H_n) \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k(3x)^k} - 2 \sum_{n=1}^{\infty} \frac{x^{n+1}}{2n+1} \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k(3x)^k} \tag{15}$$

$$\frac{\pi^2}{9} = \sum_{n=1}^{\infty} 2^{-n} \left(2H_{2n} - H_n - \frac{2}{2n+1} \right) \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} \left(\frac{2}{3} \right)^k \tag{16}$$

Entry 8. for $1/3 < x < 1$, we have

$$\frac{\pi^2}{36} = \frac{1}{1-x} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}((3x+1)n+3x)}{n(n+1)(3x)^{n+1}} \sum_{k=1}^{n} \frac{x^k - x^{n+1}}{2k-1} \tag{17}$$

$$\frac{\pi^2}{24} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(5n+3)}{n(n+1)} \left(\frac{2}{3} \right)^n \sum_{k=1}^{n} \frac{2^{-k} - 2^{-n-1}}{2k-1} \tag{18}$$

Entry 9.

$$\frac{\pi^2}{18} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{3^n n} (2H_{2n-1} - H_{n-1}) \tag{19}$$

$$\frac{\pi^2}{6} = 2 - \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{3^n (n+1)} (2H_{2n+1} - H_n) \tag{20}$$

$$\frac{\pi^2}{12} = 1 - \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{3^n (n+1)} \sum_{k=0}^{n} \frac{1}{2k+1} \tag{21}$$

Entry 10.

$$\frac{\pi^2}{12} = 4 \ln\left(\frac{4}{3}\right) - \frac{1}{3} - \sum_{n=2}^{\infty} \frac{1}{2n+1} \left(1 - 3\ln\left(\frac{4}{3}\right) - \sum_{k=1}^{n-1} \frac{(-1)^{k-1}}{3^k (k+1)} \right) \tag{22}$$

Entry 11.

$$\frac{\pi^2}{6} = -1 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{3^n} \left(\frac{6 H_{2n}}{n} + \frac{H_{n+1}}{n+1} \right) \tag{23}$$

3. Endnote

Entry 12.

$$\frac{\pi^2}{36} = \sum_{n=1}^{\infty} \frac{1}{2n-1} \left(\ln\left(\frac{4}{3}\right) - \sum_{k=1}^{n-1} \frac{(-1)^{k-1}}{k\, 3^k} \right) \tag{24}$$

$$\frac{\pi^2}{18} = \frac{\ln 2}{2} - \frac{3}{2} \ln\left(\frac{9}{8}\right) + \sum_{n=1}^{\infty} \left(\frac{2\,(-1)^{n-1}}{n\, 3^n} + \frac{1}{2n\, 3^{2n}} - \frac{3}{(2n-1)\, 3^{2n}} \right) H_{2n} \tag{25}$$

$$\frac{\pi^2}{18} = \frac{1}{4} \operatorname{Li}_2\left(\frac{1}{9}\right) - \operatorname{Li}_2\left(-\frac{1}{3}\right) + \sum_{n=1}^{\infty} \left(\frac{2\,(-1)^{n-1}}{n\, 3^n} + \frac{1}{2n\, 3^{2n}} - \frac{3}{(2n-1)\, 3^{2n}} \right) H_{2n-1} \tag{26}$$

Remark: $\operatorname{Li}_2(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2}$, $|x| < 1$ is the Polylogarithm function.

4. References

[1] Ji-Cai Liu, On two congruences involving Franel numbers, arXiv: 2002.03650v1 [math.NT] 10 Feb 2020.

[2] S. Mattarei, and R. Tauraso, Congruences for central binomial sums and finite polylogarithms, J. Number Theory 133, 2013.

[3] C. Schneider, Symbolic summation assists combinatorics, Sém. Lothar. Combin. 56 , 2007.

[4] V. Strehl, Binomial sums and identities, Maple Technical Newsletter 10, 1993.