# A proof of the Collatz conjecture

Xingyuan Zhang

(Independent scholar, Suzhou P. R. China, e-mail:502553424@qq.com)

**Abstract:** The Collatz conjecture is valid. By constructing the concepts of path number, similar number and source number, and so on, it makes in this paper a detailed analysis of the properties of the forward and inverse operations of the proposition, and finally gives an elementary proof of the conjecture. The conclusions are as follows: 1, For any given odd number $n$ , the unique forward operation path number $p$ can be obtained after one time of operation according to the propositional rules, $p$ is not a triple; to do forward operations continuously, the path number $p$ either tends towards infinity or regresses to 1; 2, To analyze the inverse operations of the proposition, for any given odd number $p$ , the inverse path number $n$ is not only obtainable, but there are an infinite number of them after one time of operation, and all of them are similar numbers each other, so that, if we do a reverse operation (called reverse tracing) , the result is that we can obtain a set of similar number; any odd number can generate a set of similar number; by repeatedly taking the similar numbers of an odd number for reverse tracing, an infinite number of tracing paths can be obtained, and on any tracing path, the odd numbers must tend to infinity; 3, It can be proved that any odd number can be obtained by successive reverse tracing starting from 1; 4, For any odd number obtained by reverse tracing, to do forward operations, it must return to 1 along the reverse tracing path.

**Keywords:** conjecture, path number, similarity number, source number, reverse tracing

**0 Introductions**

The Collatz conjecture is the 3x+1 conjecture, also known as Kakutani's conjecture. It has not been proved since it was proposed [1]. Its operation rules are: for any given positive integer, if it is an even, divide by 2; if it is an odd, multiply by 3, and add 1, and then divide by 2, until it becomes an odd again. If doing operations repeatedly it will eventually return to 1.

In this paper, we call Collatz conjecture as Collatz proposition, or proposition for short.

According to the operation rules, an even number will be transformed into an odd number firstly, so we take odd numbers directly to analysis and study for the operation. By constructing the concepts of path number, similarity number and source number, we analysis the principle of forward and reverse operations of this proposition, and draw some relevant conclusions.

**1 Operation rules of the proposition and analysis of the properties**

Here the operation rules can be described as: to take an odd number, multiplies it with 3, pluses 1, and then get an even number, divided this even number by 2, one or more times, then it becomes a new odd number; doing operations above repeatedly on these new odd numbers, Collatz conjectured that the new odd number obtained in the end must be 1(1 is cyclic, stop operation if it returns to 1).

In operations, a series of new odd numbers form an operation path. Obviously, because the new odd numbers are different, the odd number either goes back to 1 or to infinity at the end. By using a computer, a limited range of numbers (about 14 bits) is checked one by one, and they all go back to 1.

**Definition 1** (a) The operation process from an odd number to a new odd number is called one time of operation, or one operation for short; times of operations are called continuous operations; in one operation, divided by 2 is called a local operation;

(b) The new odd number obtained after one operation is called a path number of operation, or a path number for short; for two path numbers obtained in turn, it´s called mutually previous or next path number;

(c) One operation done in the order of the proposition is called one time of forward operation.

Next, we give the formula and analyze the properties of forward operation.

### 1.1 The formula

For any given odd number $n$, let $p$ be its path number, then according to the operation rules, we have

$$p = \frac{3n+1}{2^k} \tag{1}$$

Where $k \in N$ and $2^k$ is a divisor.

Here, we called formula (1) as the forward operation formula of the proposition. Analysis Formula(1) , it is not difficult to obtain: for any given odd number $n$, there has only one path number $p$ corresponding to $n$; the value of $k$ is determined by the odd number $n$, $k$ can be expressed as the times divided by 2, for example, when it equals to 1, that it means in one operation, there is one local operation divided by 2; when it equals to 3, there are three local operations divided by 2; for two different odd numbers, the times divided by 2 are the same or different because the values of $k$ in the divisor can take all of the positive integers, therefore, perhaps there are infinite odd numbers that they will all get the same path number after one operation, and these odd numbers have some correlation properties with each other.

### 1.2 Numerical comparison of $n$ and $p$

(a) As can be seen from formula (1), we can get

If, $k = 1 \quad p > n$

If, $k \geq 2 \quad p < n$

Obviously, the larger $k$ is, the greater the change rate of the path number is (to reduce).

(b) Changes of the value of two adjacent numbers

Let $n$ be an odd number and expressed as $2x+1$, where $x = 0$ or $x \in N$.

Thus, one of its adjacent odd numbers can be expressed as

$$2x+1+2.$$

To do one operation for $2x+1$, then we get

$$\frac{3(2x+1)+1}{2} = 3x+2 \cdot$$

To do one operation for $2x+1+2$, then we get

$$\frac{3(2x+1+2)+1}{2} = 3x+5 \cdot$$

Obviously, in these two numbers above, one is odd and the other is even. They both increase firstly. Since the even number can be divided by 2 again, so it will decrease finally. From this, we can get a conclusion (conclusion (1)): for two adjacent numbers, the path numbers of them if one becomes larger, the other must becomes smaller.

### 1.3 The properties of the path numbers

An odd number $n$ can be expressed as

$$n = 2x+1.$$

Where $x = 0$ or $x \in N$.

To do one operation for $n$, suppose we can get a path number $3p$, where $p$ is an odd number, then from Formula (1), we have the following equation

$$3p = \frac{3(2x+1)+1}{2^k} = \frac{6x+4}{2^k}.$$

To simply, then we have

$$x + \frac{2}{3} = 2^{k-1} p.$$

Obviously, the equation above does not hold for integers, so we can get the following conclusion (conclusion (2)): the path number $p$ is not a triple, but a non-triple.

**1.4 narrow paths**

**Definition 2** Let $n$ be an odd number, to do operations continuously for it, in every operation process, the times of local operation divided by 2 does not exceed 2, i.e. $k \le 2$, thus we called a section of the path as a narrow path which is composed of the odd number and its path numbers.

Obviously, on the narrow path, the numerical change rate of the path numbers is the smallest, that is, the range of change is the narrowest.

**2 The similar numbers and their properties**

From the analysis at 1.1, it's known that the same path number can be obtained when doing one operation for two odd numbers respectively. For example, if doing one operation for 7 and 29, they both get 11. For 7, the value of $k$ in Formula (1) is 1, and for 29, the value of $k$ is 3.

**Definition 3** Suppose, there are two odd numbers $n_1$ and $n_2$ whose path numbers are the same of $p$, then, we called that $n_1$ is a similar number of $n_2$, or $n_2$ is a similar number of $n_1$, that is, they are similar each other; and denoted $n_1 \backsim n_2$, or, $n_2 \backsim n_1$.

For example, 29 is a similar number of 7, or 7 is a similar number of 29.

Obviously, the similar numbers are caused by different values of $k$.

Next, we analyses the properties of similar numbers.

**2.1 The relationship between similar numbers**

Suppose, there are two similar numbers $n_1$ and $n_2$, where $n_2 > n_1$, doing one operation on each of them, we can get the path numbers $p_1$ and $p_2$. According to formula (1), we have

$$p_1 = \frac{3n_1 + 1}{2^{k_1}}$$

and

$$p_2 = \frac{3n_2 + 1}{2^{k_2}}.$$

Where $k_1 \in N$, and $k_2 \in N$.

Now, let $p_1 = p_2$, then we have

$$\frac{3n_1+1}{2^{k_1}}=\frac{3n_2+1}{2^{k_2}}.$$

Since $n_2 > n_1$, so, $k_2 > k_1$. From the upper equation, $n_2$ can be obtained, that is

$$n_2=\frac{2^{k_2}}{2^{k_1}}n_1+\frac{1}{3}\left(\frac{2^{k_2}}{2^{k_1}}-1\right)=2^{k_2-k_1}n_1+\frac{1}{3}\left(2^{k_2-k_1}-1\right).$$

Obviously, $k_2-k_1$ are positive integers in the equation above; if $n_2$ to be an integer, $2^{k_2-k_1}-1$ must be a triple, it has a minimum value of 3, that is, $2^{k_2-k_1}$ is 4.

Now, let $k_2-k_1=2k$, i.e., $k_2-k_1$ takes even numbers, where $k\in N$, so we have

$$2^{k_2-k_1}-1=2^{2k}-1=(2^k+1)(2^k-1).$$

As it can be seen that there is a triple in the three continuous numbers of $2^k-1$, $2^k$ and $2^k+1$; and that when $k_2-k_1=2k+1$, i.e., $k_2-k_1$ takes odd numbers, $2^{2k+1}-1$ has not any triple factor. Thus, $k_2-k_1$ must take even numbers, that is, $2^{k_2-k_1}$ must takes 4 or 4 times of 4, and then $n_2$ has an integer solution in the equation above.

Next, we analyze some cases with 4 and its 4 multiples respectively,

1) when $2^{k_2-k_1}$ takes 4, we have

$$n_2=4n_1+1$$

2) when $2^{k_2-k_1}$ takes 16, we have

$$n_2=16n_1+5=4(4n_1+1)+1$$

3) when $2^{k_2-k_1}$ takes 64, we have

$$n_2=64n_1+21=4[4(4n_1+1)+1]+1.$$

As we can see, the formula above is an iterative formula. It follows that when the gap between two similar numbers is smallest, we obtain

$$n_2=4n_1+1 \qquad (2)$$

Here, we called formula (2) as the formula of the similar numbers, and also, two similar numbers when they have the smallest difference between them as the adjacent similar numbers. By using this formula, we can find out the numberless similar numbers of $n_1$ in turn.

This relationship can be verified by doing operations on $n_1$ and $n_2$ separately.

1) For $n_1$, we have

$$\frac{3n_1+1}{2^k}$$

2) For $n_2$, we have

$$\frac{3(4n_1+1)+1}{2^k}=4\left(\frac{3n_1+1}{2^k}\right).$$

Obviously, the number on the right can be divided by 4 again, thus we also have

$$\frac{3n_1+1}{2^k}.$$

As it can be seen that when we do operations on $n_1$ and $n_2$ respectively, we get the same number, so they are similar numbers to each other.

Doing inverse operation for formula (2), then we have

$$n_1=\frac{n_2-1}{4} \tag{3}$$

Obviously, if $n_1$ is an integer, then it is an adjacent similar number of $n_2$, and $n_1 < n_2$. Here Formula (3) is called the inverse operation formula of similar numbers.

**2.2 The set of similar numbers**

It is obvious from formula (2) that any odd number can generate an infinite number of similar numbers in turn.

**Definition 4** (a) Suppose, the similar numbers generated by odd number $n$ in turn are $n_1, n_2, \cdots, n_i$, where $i \in N$, $n_1 > n$, $n_{i+1} > n_i$, here, we called the infinite set composed of $n$ and $n_i$ as a infinite set of similar numbers, or a similar number set for short;

(b) We called $n$ and $n_i$ the previous similar number of $n_1$ and $n_{i+1}$ respectively, called $n_1$ and $n_{i+1}$ as the next or backward similar number of $n$ and $n_i$ respectively;

(c) We called $n$ as the generating number of the set (the first, also the smallest similar number in the set); called the set as number $n$ set.

For examples, 1 can generate 5, 21, 85, 341, 1365⋯an infinite number of similar numbers, 1 and all its similar numbers constitute a set of similar numbers, 1 is the generating number of the set, i.e., the first and the smallest similar number; this set is called number 1 set, in which any similar number returns directly to 1 after one operation. In the same way, 3 can generate similar numbers such as 13, 53, 213, 853, and so on, they constitute the number 3 set. In a set of similar numbers, all numbers constitute a sequence. Specifically, if every number in number 1 set be added 3 on the right in turn, then number 1 set becomes number 3 set.

### 2.3 The smallest gap between two triples in a similar number set

In a continuous series of odd numbers, two adjacent triples are separated by two non-triples. 3 is the smallest triple, and 3 added by 6 every time, it becomes another triple (odd number). Similarly, in a similar number set, two adjacent triples are separated by two non-triples.

To verify as follows:

A triple can be expressed as $3n$, where $n$ is an odd number, so according to Formula (2) , the following three similar numbers are as follows in turn:

The first $4(3n)+1=12n+1$

The second $4(12n+1)+1=48n+5$

The third $4(48n+5)+1=192n+21=3(64n+7)$

As it can be seen, the first and the second number are not triples, and the third number is exactly a triple.

### 2.4 The effect of the similar numbers in operations

In the continuous operations, when a path number has a similar number smaller than itself, the next path number will quickly become smaller, or even, directly back to 1. For example, the number 853 is similar to 3, which returns to 5 quickly after one operation; the number 1365 is similar to 1, which returns to 1 directly after one operation. Thus, those numbers are the ending-numbers of a narrow path. It is not difficult to see that the similarity existing in odd numbers is of great significance in judging this proposition.

## 3 Analysis of the principle of the reverse operations

The forward operation of the proposition is reversible for non-triples. Now, to do a reverse operation for Formula (1), then we have

$$n=\frac{2^k p-1}{3} \tag{4}$$

or

$$3n=2^k p-1 \tag{4-1}$$

Where $k \in N$ , and $p$ takes non-triples (conclusion (2)).

Here, Formula (4) or (4-1) is called the inverse formula of the proposition, $2^k$ is called a multiplier, and $n$ is called the inverse path number. The formulas are used in reverse to find the inverse path number $n$ of $p$ obtained in the forward operation.

By analyzing in section 1.1, we know that there are an infinite number of inverse path numbers, each of them constitutes the source of the known path number $p$ , i.e., $p$ is from the infinite of inverse path numbers.

Next, we analyze the properties of the inverse operations and draw some conclusions.

### 3.1 The relationship between the inverse path numbers and the properties of the minimum inverse path number

#### 3.1.1 The relationship between the inverse path numbers

From formula (4), it can be seen that the inverse path number is directly related to the value of $k$ in the multiplier $2^k$, and therefore, we use the parity property of the values of $k$ to analyze the relationship between the inverse path numbers. Apparently, for any non-triple $p$, we can't get an integer at the same time when we take the minimum odd number 1 and the minimum even number 2; for $k$ and $k+2$, it's the same as an odd or even number.

Let $p$ be a non-triple, $n$ be an inverse path number, according to Formula (4), then we have

$$n=\frac{2^k p-1}{3}.$$

Multiplies 4 for the two sides above, then we obtain

$$4n=4\left(\frac{2^k p-1}{3}\right)=\frac{2^{k+2} p-4}{3}=\frac{2^{k+2} p-1}{3}-1.$$

That is

$$4n+1=\frac{2^{k+2} p-1}{3}.$$

Hence, comparing with the similar number formula (formula (2)), it is not difficult to see, that the right of the equation above is the next similar number of $n$, marked $n_1$, it's generated by adding 2 to $k$ in the multiplier, that is, when $k$ takes the next odd or even number, we can get the next similar number $n_1$ of $n$. In the same way, when $k$ takes again next odd or even number, we can get the second similar number of $n$, marked $n_2$. Obviously, as the value of $k$ increases, we can get an infinite number of similar numbers of $n$.

Now, the analysis above shows that, for the same inverse path number $n$, $k$ is either odd or even. When taking the minimum odd number 1, the minimum multiplier is 2; when taking the minimum even number 2, the minimum multiplier is 4.

Now, a conclusion (conclusion 3) can be drawn by the analysis above: In formula (4), for a non-triple $p$, if $k$ takes the smallest odd number 1(multiplier is 2), we can get an inverse path number, then $k$ takes the rest odd numbers greater than 1, we can also get an infinite of numbers of inverse path numbers, and all the inverse path numbers are similar to each other; if, when $k$ takes the smallest odd number 1, we can't get an inverse path number, then $k$ takes the smallest even number 2(multiplier is 4) and any even number greater than 2, we can get an infinite of numbers of inverse path numbers definitely, and they're also similar numbers each other.

When $k$ takes the smallest odd or even number, i.e., multiplier $2^k$ takes the smallest 2 or 4, the inverse path number is called here the smallest inverse path number.

#### 3.1.2 The properties of the smallest inverse path numbers

Suppose, there are two adjacent similar numbers $n_1$ and $n_2$ in a set, where $n_2$ is known, and $n_2 > n_1$, i.e., $n_1$ is the previous similar number of $n_2$, according to formula (4), then we have

$$n_2=\frac{2^k p-1}{3}.$$

According to Formula (3), we have

$$n_1 = \frac{n_2 - 1}{4}.$$

To substitute $n_2$ above, then we have

$$n_1 = \frac{\frac{2^k p - 1}{3} - 1}{4} = \frac{2^k p - 1 - 3}{12} = \frac{2^k p}{12} - \frac{1}{3}.$$

Next, we take multiplier 2 and 4 for analysis respectively.

1）when taking 2

$$n_1 = \frac{2p}{12} - \frac{1}{3} = \frac{p}{6} - \frac{1}{3} = \frac{p-2}{6},$$

2）when taking 4

$$n_1 = \frac{4p}{12} - \frac{1}{3} = \frac{p}{3} - \frac{1}{3} = \frac{p-1}{3}.$$

Obviously, $p-2$ is an odd number, $p-1$ is an even number, so, neither of these two equations above can get an integer, thus, there is no similar number less than $n_2$, that is, $n_1$ does not exist. Therefore, it can be concluded (conclusion (4)): the smallest inverse path number must be the generating number of a similar number set (i.e. $n_2$ is the first one in a set).

**Definition 5** (a) When doing the reverse operations according to Formula (4), we change the names, called the given originally non-triple $p$ as a primitive number; called the smallest inverse path number as a source number of the primitive number $p$ ;

(b) Doing one time of reverse operation according to Formula (4) is called one time of reverse operation tracing, tracing for short; times of tracing is called continuous tracing.

Obviously, since the rest of the inverse path numbers of a primitive number are similar to the source number (according to conclusion (3)) , they can be found in turn by way of the formula of similar numbers (formula (2)) , and therefore, here we defined only the first of these.

### 3.2 Analysis of the multiplier

#### 3.2.1 analysis of the multiplier of two continuous primitive numbers

As stated in 2.3, for a continuous series of odd numbers, there are only two consecutive non-triples between two adjacent triples, i.e., two consecutive primitives.

Let $3p$ be a triple, where $p$ is an odd number, so, in terms of the increasing value, the first primitive number adjacent to $3p$ is $3p+2$, and the second is $3p+4$. Next, we take the multiplier 2 and 4 for analysis respectively.

##### 3.2.1.1 Take 2

According to the formula (4-1), we have

$$3n = 2p - 1. \qquad \text{(4-1-1)}$$

a) Put the first primitive number $3p+2$ into the formula (4-1-1), and then we have

$$3n = 2(3p+2) - 1 = 6p + 3.$$

That is

$$n = 2p + 1.$$

Obviously, there are integer solutions to the equation, and they are source numbers of first primitive number.

b) Put the second primitive number $3p+4$ into the formula (4-1-1), and then we have

$$3n = 2(3p+4) - 1 = 6p + 6 + 1.$$

That is

$$n = 2p + 2 + \frac{1}{3}.$$

Obviously, the equation has no integer solution.

**3.2.1.2 Take 4**

According to the formula (4-1), we have

$$3n = 4p - 1. \qquad \text{(4-1-2)}$$

a) put the first primitive number $3p+2$ into the formula (4-1-2) , then we have

$$3n = 4(3p+2) - 1 = 12p + 6 + 1.$$

That is

$$n = 4p + 2 + \frac{1}{3}.$$

Obviously, the equation above has no integer solution.

b) put the second primitive number $3p+4$ into the formula (4-1-2) , then we have

$$3n = 4(3p+4) - 1 = 12p + 15.$$

That is

$$n = 4p + 5.$$

Obviously, there are integer solutions to the equation, and they are source numbers of the second primitive number.

Thus, based on the analysis of 3.2.1.1 and 3.2.1.2, it can be concluded (conclusion (5): in the continuous series of odd numbers, for two continuous primitive numbers, the multiplier of the first primitive number is taken 2 and the multiplier of the second is taken 4 definitely. And as a result, their source numbers, one gets smaller and another gets larger (being similar to conclusion (1)).

Here, we called the formula (4-1-1) and (4-1-2) as the formulas of source numbers.

**3.2.2 Analysis of the multiplier of any primitive number in the series of continuous odd numbers**

**3.2.2.1 Set the multiplier of a triple**

Since formula (4) does not hold for the triples, therefore, the multipliers of triples is set as 0, which means that there is no source number for the triples. Thus, for two consecutive primitive numbers and a triple, i.e. three consecutive odd numbers, as derived from conclusion (5), their multipliers in order are 2, 4 and 0.

### 3.2.2.2 A special determinant of odd numbers

According to the conclusion (5), the continuous odd numbers can be arranged in a special determinant in tabular form, and then the special regular of multipliers can be shown. See Table 1 Table of multipliers. Next, we use this table to analyze the method for taking the multipliers. In the table, the multipliers for each row are repeated four times in the order of 2, 4 and 0.

Tab.1 Table of multiplier ( $2^k$ )

| row | con- | column $l$ | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $h$ | tent | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
| | $2^k$ | **2** | **4** | **0** | **2** | **4** | **0** | **2** | **4** | **0** | **2** | **4** | **0** |
| 1 | | | | | | | | | | | | 1 | 3 |
| 2 | | 5 | 7 | 9 | 11 | 13 | 15 | 17 | 19 | 21 | 23 | 25 | 27 |
| 3 | | 29 | 31 | 33 | 35 | 37 | 39 | 41 | 43 | 45 | 47 | 49 | 51 |
| 4 | odd | 53 | 55 | 57 | 59 | 61 | 63 | 65 | 67 | 69 | 71 | 73 | 75 |
| 5 | num. | 77 | 79 | 81 | 83 | 85 | 87 | 89 | 91 | 93 | 95 | 97 | 99 |
| 6 | | 101 | 103 | 105 | 107 | 109 | 111 | 113 | 115 | 117 | 119 | 121 | 123 |
| 7 | | 125 | 127 | 129 | 131 | 133 | 135 | 137 | 139 | 141 | 143 | 145 | 147 |
| 8 | | 149 | 151 | 153 | 155 | 157 | 159 | 161 | 163 | 165 | 167 | 169 | 171 |
| ... | | | | | | | | | | | | | |
| $h$ | | $n$ | | | | | | | | | | | |
| notes | 1. This table shows consecutive odd numbers in rows and columns; there are 1 and 3 in first row and the multipliers of them are corresponding to 4 and 0; from second row, there are twelve odd numbers in each, and the multipliers of them are corresponding to 2, 4 and 0 four times in order; 2. Characters of the triples are with shadows, and they form four columns, the remaining 8 columns are primitive numbers. | | | | | | | | | | | | |

### 3.2.2.3 Location analysis of an odd number in Tab. 1

a） Row number $h$

Let $n$ be an odd number given arbitrarily, then, according to the odd number arrangement in the table, its row number $h$ is given by the following formula

$$h = \frac{n-4}{24} + 1 \tag{5}$$

Where, $h$ takes an integer approach a large number.

Here, we called formula (5) the row number formula.

b）Column number $l$

Let the row number $h$ be known, and then the column number $l$ of the odd number $n$ in the table is given by the following formula

$$l = \frac{n-4-24(h-2)}{2} \tag{6}$$

Where, $l$ takes an integer approach a large number.

Here, we called formula (6) the column number formula.

Obviously, according to the column number $l$ in the table, we can cross-reference to get the multiplier, that is,

When $l$ equals to 1, 4, 7, 10, the odd number is a primitive number, and its multiplier is 2;

When $l$ equals to 2, 5, 8, 11, the odd number is a primitive number, and its multiplier is 4;

When $l$ equals to 3, 6, 9, 12, the odd number is a triple, its multiplier is 0.

c）The first odd number in the row of $h$

If, $h$ is known, then the first odd number $n$ in the row of $h$, is given by the following formula

$$n = 5 + 24(h-2) \tag{7}$$

Where, $h \geq 2$

In this row, the rest numbers increase by 2 in turn. Obviously, we can also get a multiplier based on the number of increments by 2.

Now, by analysis of 3.2.1 and 3.2.2, it can be concluded (conclusion (6)): For any given odd number (a primitive number or a triple), its row number $h$ and column number $l$ in the table can be found, thus, we can determine the multiplies (in which, multiplier 0 does not make sense, corresponding to the multiplier 2 or 4, the odd number is a primitive number). According to the multiplier (2 or 4) of the primitive number, the source number of the primitive number can be obtained by using formula ((4-1-1) or (4-1-2)).

Here, the specific order of which a source number is calculated is as bellows again:

a） by using the row number formula (formula (5)) , the row number $h$ of an odd number in the table can be found out;

b） by using the column number formula (formula (6)), the column number $l$ of an odd number in the table can be found out;

c） according to the column number $l$, to determine if the odd number is a primitive number or not, if it is, selecting its multiplier;

d） And then, by using the selected multiplier, to find the source number of this primitive number according to the formula of source numbers (formula (4-1-1) or (4-1-2)).

As it can be seen, the multiplier can be determined according to the primitive number itself. Thus, the source number of any primitive number can be found out, and an infinite number of similar numbers of any source number can also be found out by using the formula of similar number (formula (2)).

**3.2.3 A simple method for determining the multiplier**

Since the multiplier is either 2 or 4, thus, for any given odd number $p$, here, $p$ perhaps be a primitive number or a triple, there is a simple method. Firstly, we use 2 directly in formula (4) to try to find out the source number $n$. If $n$ is an integer, so that, 2 is its multiplier, and $p$ is a primitive number, the integer $n$ is its source number. If $n$ is not an integer, then use 4 to try secondly; if neither of $n$ is an integer, then $p$ must be a triple, and it has no source number.

**3.3 Analysis of tracing and continuous tracing path**

According to conclusion (6), source number $n$ can be obtained by tracing for primitive number $p$. Obviously, if $n$ is not a triple, then $n$ can be regarded as another primitive number for tracing. Over and over again, primitive number $p$ with one or more of its source numbers can form a continuous tracing path.

Next, we analyze the properties of continuous tracing paths.

**3.3.1 The end of a continuous tracing path**

Since a triple has no source numbers, so a continuous tracing path ended at a triple. For some special primitive numbers, their source numbers are triples getting from formula (4-1-1) or formula (4-1-2).

**3.3.2 Integrity of the continuous tracing path**

If the primitive number $p$ has the property of arbitrary selection, then it may itself be a minimum inverse path number, that is, it is also a source number, although its source numbers can form a continuous tracing path, but it is not complete. So, at this time, it ´s need to do some continuous forward operations on this primitive number until getting a forward operation path number, when doing forward operation on it, it has more than two local operations divided by 2, that is, it has a similar number less than itself.

A complete tracing path begins with a primitive number which has a similar number less than itself, and ends at a triple. In this path, every number except the primitive number is a source number. Obviously, in the opposite direction, that is, the forward path from the final source number to the primitive number is a narrow path.

Those complete successive tracing paths have different length, that is, for different primitive numbers, the quantity of source numbers varies. For examples, tracing for 1, 1 can be traced itself (its multiplier is 4, a forward operation on it also yields itself, 1 is a special odd number); Tracing for the similar number 5 of 1, we get the minimum triple 3, so the path has only two odd numbers 5 and 3; For 35, 23 and triple 15 are obtained by successive tracing twice, while for 445, tracings of 17 times are required to obtain the triple 27, there are 18 odd numbers in this path.

**Definition 6** If a primitive number $p$ and its source numbers, constitute a complete continuous tracing path, then we called the set composed by the primitive number $p$ and its source numbers as a source number set.

**3.3.3 The transition between two complete successive tracing paths or two sets**

As defined above, tracing for a primitive number can constitute a complete continuous tracing path and form a source number set. Now, take the next similar number of the primitive number (if it's a triple, to skip it, take the third) for tracing continuously. Obviously, this similar number as a primitive number, it can also generate a complete continuous tracing path and forms a source number set. At this time, the similar number becomes the transition number between the two complete tracing paths, that is, it acts as an operational link between two sets. If the source numbers in two sets are arranged upwards respectively, then they assume a U-shaped structure (the similar symbols '$\backsim$' acts as a link at the bottom).

According to the conclusion (4) and definition 5, each source number in a set can generate a new set of similar numbers as a generating number. Therefore, if we take the second similar number in a new set (if it is a triple, to skip it and take the third one) for continuous tracing, we can also get a new set of source numbers. This new set and the previous set in which the generating number is located have a similar h-shaped structure (the source numbers in two sets are also arranged upwards respectively, the similar symbols '$\backsim$' acts as a link in the middle).

Obviously, by repeatedly taking similar numbers for tracing, more and more sets can be obtained.

**3.3.4 The tendency of the continuous tracing paths**

Continuous reverse tracing is carried out continuously by the combination of the two operations of finding source numbers and similar numbers. In the process of tracing, the similar numbers are increasing, on the contrary, that is, if doing forward operations continuously, the similar numbers are decreasing, therefore, it is not difficult to draw a conclusion (conclusion (7)): by tracing continuously for a given odd number, there will be an infinite number of paths and each of them tends to large numbers, that is, different maximum odd numbers must appear repeatedly in the path; but on the contrary, paths tend to the small numbers, that is, different minimum odd numbers must appear repeatedly in the path.

Obviously, continuous tracing can start from 1. On the one hand, we find out the similar numbers of 1, such as 5, 21, 85, 341, and so on, on the other hand, we find out the source number 3 of 5, and next, we find out the similar numbers of 3, such as 13, 53, 213, 853, and so on. And next again, we are to find the source numbers of 85 and 13 respectively; continuous tracing in this way. To do forward operations for any of these odd numbers, it will return to 1 definitely.

For a primitive number $p$ , if keep tracing, there will be more and more paths, and for one of them, as an example, the triple path, it can be described as: tracing for a primitive number $p$ , then get a number of source numbers in turn, among them, the last is a triple, take the next similar number of this triple (it is not a triple definitively, see 2.3) for tracing, and then get some source numbers again of which the last is still a triple. And so over and over again, the triple is going to approach infinity eventually. This triple path is illustrated in the following figure.

**3.3.5 An example of continuous tracing path and related explanations**

For the odd number 27, if doing forward operations, it returns to 1. Comparatively speaking, this number needs more operations than any other two-digit numbers. Therefore, the odd number 27 is taken as an example to illustrate several successive tracing paths and to demonstrate the principle of reverse operations. See Fig.1 Reverse tracing path graph of odd number 27.

As we can see, starting from 1 at the lower left-hand corner and going through 8 times of similar numbers transitions, it ends up with the triple 27 at the top. The legend and path are explained as follows:

a) This graph depicts the reverse path from 1 to 27, which is also the forward path from 27 to 1;

b) There are some similar numbers of a set in the horizontal direction, and the similar symbols '∽' are used as the connected characters between the adjacent similar numbers; five similar numbers of 1 is listed, and the triples be with shadows;

c) There are source number sets in the vertical direction; the symbol ' ↓ ' indicates 'sourced from', limited to page height, there is a little, only at the top of 5, 13, 37, 229, 541 and 5125; for a set, tracing from the primitive number at the bottom, getting the source numbers in order up and the path ended at a triple on the top; the source numbers and the primitive number form a complete narrow path;

d) The triples at the top appear to be on the path, but they are skipped when doing forward operations. For example, doing a single operation on 445 leads directly to 167, thus the triple 111 is skipped; as the same, the triples in the set of similar numbers are also skipped directly, that is why the forward operations do not yield any triple;

e) In the graph, the red path is the triple path which is derived from the triples at the top and the next similar numbers of them;

f) To take the similar number 109 of 27 for tracing continuously, there will be more and more paths, spreading out like branches of a tree, and the following paths will all get longer and longer, and finally, each of them tends to infinity.

| | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | | | | | | | | | | 27 |
| | | | | | | | | | | | | | | | | 41 |
| | | | | | | | | | | | | | | | | 31 |
| | | | | | | | | | | | | | | | | 47 |
| | | | | | | | | | | | | | | | | 71 |
| | | | | | | | | | | | | | | | | 107 |
| | | | | | | | | | | | | | | | | 161 |
| | | | | | | | | | | | | | | | | 121 |
| | | | | | | | | | | | | | | | | 91 |
| | | | | | | | | | | | | | | | | 137 |
| | | | | | | | | | | | | | | | | 103 |
| | | | | | | | | | | | | | | | | 155 |
| | | | | | | | | | | | | | | | | 233 |
| | | | | | | | | | | | | | | | | 175 |
| | | | | | | | | | | | | 8097 | ∽ | | | 263 |
| | | | | | | | | | | | | 6073 | | | | 395 |
| | | | | | | | | | | | | 4555 | | | | 593 |
| | | | | | | | | | | | | 6833 | | 111 | ∽ | 445 |
| | | | | | | | | | | | | ↓ | | 167 | | |
| | | | | | | | | | | 1281 | ∽ | 5125 | | 251 | | |
| | | | | | | | | | | 961 | | | | 377 | | |
| | | | | | | | | | | 721 | | | | 283 | | |
| | | | | | | | | | | ↓ | | | | 425 | | |
| | | | | | | | | 135 | ∽ | 541 | | 159 | | 319 | | |
| | | | | | | | | 203 | | | | 239 | | 479 | | |
| | | | | | | | | 305 | | | | 359 | | 719 | | |
| | | | | | | | | ↓ | | | | 533 | | 1079 | | |
| | | | | | | 57 | ∽ | 229 | | | | 809 | | 1619 | | |
| | | | | | | 43 | | | | 303 | | 607 | ∽ | 2429 | | |
| | | | | | | 65 | | | | 455 | | 911 | | | | |
| | | | | | | 49 | | | | 683 | | 1367 | | | | |
| | | | | | | ↓ | | | | 1025 | | 2051 | | | | |
| | | | | 9 | ∽ | 37 | | | | 769 | ∽ | 3077 | | | | |
| | | | | 7 | | | | | | 577 | | | | | | |
| | | | | 11 | | | | | | 433 | | | | | | |
| | | | | 17 | | | | 81 | ∽ | 325 | | | | | | |
| | | | | ↓ | | 15 | ∽ | 61 | | | | | | | | |
| | | | | ↓ | | 23 | | | | | | | | | | |
| | | | | ↓ | | 35 | | | | | | | | | | |
| | | 3 | ∽ | 13 | ∽ | 53 | | | | | | | | | | |
| | | ↓ | | | | | | | | | | | | | | |
| 1 | ∽ | 5 | ∽ | 21 | ∽ | 85 | ∽ | 341 | ∽ | 1365 | ∽ | | | | | |

Fig.1 Reverse tracing path graph of odd number 27

## 4 Analysis of density of the odd numbers and the final conclusion

As stated in section 1, for any given odd number $n$, the forward path number is either going back to 1 or tending towards infinity.

Now, we analysis the density of the odd numbers obtained by successive tracing.

Suppose, there is an odd number $n$ that it has not been traced on the paths starting from 1, that is, $n$ has been missed. It is obvious that we can do continuous inverse and forward operations on it. When doing inverse operations continuously, the inverse path numbers must tend to infinity (conclusion (7)), when doing forward operations continuously, the forward path numbers must also tend to infinity, because if its forward path numbers get smaller, it must eventually reach 1, it shows that there must be a reverse tracing path between 1 and $n$. From this, for $n$ both positive and negative operations tend to infinity. However, the inverse operations are just in the opposite direction based on the same kind of operation rules of the forward direction. Therefore, the assumption above does not hold, and the odd number $n$ must not only be in the range of the odd numbers obtained by tracing, but it must also be regressed to 1. Thus it can be seen that any odd number can be obtained by successive inverse tracing stating from 1, that is, corresponding to the natural number axis, the density of the odd numbers which are obtained by inverse tracing must be 1/2(1 is also included as the starting point).

From this analysis, it can be concluded (conclusion (8)): finally, for any natural number (an even number is transformed into an odd number firstly), to do forward operations, it must follow the inverse paths analyzed in this paper and return to 1. So, the Collatz conjecture holds. In this paper, the basic operation principle of the proposition of the conjecture is expounded.

**Acknowledgments**

I would like to express my thanks to Liansheng Zhang for encouragement and help, who is a professor in Suzhou university of Science and Technology. I also thank the reviewers for comments.

**Conflicts of Interest**

The author declares no conflicts of interest regarding the publication of this paper.

**Reference**

[1] Guy, R.K. (2007) Unsolved Problems in Number Theory: The 3x + 1 Problem. Springer Verlag, New York, 330-336.