Cosine Integral

by

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abstract

We give some formulas related to $\gamma = 0.577215...$, here γ is the Euler's constant .

Introduction

The Euler-Mascheroni constant is defined by

$$\gamma = \lim_{n\to\infty}\left(H_n - \ln n\right) = 0.577215... \tag{1}$$

where

$$H_n = 1+\frac{1}{2}+\frac{1}{3}+...+\frac{1}{n} \quad , n\in\mathbb{N} = \{1,2,3,...\} \tag{2}$$

The Cosine integral is defined by

$$Ci(s) = -\int_s^{\infty}\frac{\cos x}{x}dx \tag{3}$$

$$Ci(s) = \gamma + \ln s + \int_0^{s}\frac{\cos x - 1}{x}dx \tag{4}$$

$$Ci(s) = \frac{1}{2}\left(Ei(is) + Ei(-is)\right) \tag{5}$$

where $Ei(x)$ is the exponential integral.

In this note we give some formulas related to $\gamma = 0.577215...$.

Formulas

Define $f(s)$ by

$$f(s) = \sqrt{1+s^2}\; e^{Ci(s)} = \sqrt{1+s^2}\;\exp\left(-\int_s^{\infty}\frac{\cos x}{x}dx\right) \quad , s>0 \tag{6}$$

Entry 1.

$$s_0 = 3.5 \quad , \quad s_{n+1} = f(s_n), n\in\mathbb{N} \Rightarrow \lim_{n\to\infty} s_n = \alpha = 3.52478987994771... \tag{7}$$

Entry 2.

$$\gamma = \int_0^{\alpha} \left(\frac{1}{x\left(1+x^2\right)} - \frac{\cos x}{x} \right) dx \tag{8}$$

$$\int_{\alpha}^{\infty} \left(\frac{1}{x\left(1+x^2\right)} - \frac{\cos x}{x} \right) dx = 0 \tag{9}$$

Entry 3.

$$\gamma = -\frac{1}{2}\ln\left(1+\alpha^2\right) + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\alpha^{2n}}{(2n)(2n)!} \tag{10}$$

Entry 4.

$$f'(s) = e^{Ci(s)} \left(\frac{s}{\sqrt{1+s^2}} + \frac{\sqrt{1+s^2}}{s} \cos s \right) \tag{11}$$

$$f'(\alpha) = -0.00196... \tag{12}$$

Entry 5.

$$\frac{1}{2}\ln\left(1+\alpha^{-2}\right) = \cos\alpha \int_0^{\infty} \frac{x e^{-\alpha x}}{1+x^2} dx - \sin\alpha \int_0^{\infty} \frac{e^{-\alpha x}}{1+x^2} dx \tag{13}$$

Entry 6.

$$\gamma = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n)(2n)!} - \frac{1}{2}\ln\left(1+a^{-2}\right) + Ci(1) - Ci(a) \tag{14}$$

References

1. D.H. Bailey and J.M. Borwein, Experimental Mathematics: examples,methods and implications. Notices Amer. Math. Soc., 52 (5): 502-524.
2. J.M. Borwein and D.H. Bailey, Mathematics by Experiment: Plausible Reasoning in the 21st century. AK Peters Ltd, Natick, MA, 2003.