# A Revised Prime Number Counting Function

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Abstract: It has been human's dream for thousands of years to find the ideal prime number counting function and prime number universal formula. For the number of prime numbers smaller than the given order of magnitude, the calculation deviation of the prime number theorem function x/ln (x) is relatively large. However, the calculation with Riemann formula R (x) function is too complex, and it deviates from the true value as Gaussian Li (x) function when the given order of magnitude is very large. Now the prime number theorem function x/ln (x) is dynamically modified and further optimized by using the calculation formula of Mersenne's prime number, and a simple, easy and relatively accurate calculation formula is found.

Key words: prime number theorem, number of prime numbers, prime number counting function, dynamic correction

As early as 2000 years ago, people found that prime numbers have special properties and are the basic units of integers. The "Original Geometry" about 300 BC contains important theorems related to prime numbers [1].

Finding the ideal prime number counting function and prime number universal formula has been the dream of human beings for thousands of years. Many famous mathematicians in history have studied the problem, from Euclid's first proof that there are infinite prime numbers, to the Euler product formula which combines prime numbers and ζ functions. Gauss discovered the prime number theorem in 1792 and Legendre discovered the same theorem in 1798 [2]. But at present, Riemann is still the mathematician who has made the greatest breakthrough in the prime number theory. Riemann has made an unprecedented exposition on the distribution of prime numbers [3].

Riemann function R(x) for calculating the number of prime numbers needs to apply the calculation of non-trivial zeros of Riemann ζ function, which is complicated. The calculation of Riemann function R(x) deviates from the true value as Li(x) function when the given order of magnitude is very large [3].

Prime numbers are randomly distributed on the number axis. A large number of data show that the probability of the existence of prime numbers in adjacent regions has some degree of similarity, so the distribution type of prime numbers belongs to deterministic random distribution [4]. Although the distribution of prime numbers is random, the number is certain, so it can be determined that there is a function that can relatively accurately represent the number of prime numbers. The existing data shows that with the increase of the integer value, the growth rate of the prime number gradually slows down. Therefore, this paper selects the calculation formula of Mersenne's prime number[5], which has a slow growth of function value, to dynamically modify the prime number theorem function x/ln (x) , and obtains a estimation formula of prime number expressed by elementary functions. The numerical experiment results are ideal, and the function is simple and accurate.

1. The number of prime numbers less than the given order of magnitude

The deviation of calculation value of the prime number theorem function x/ln (x) is large and always smaller than the actual prime number count. Many mathematicians have revised it in history. There are many correction methods, some of which increase the monomial formula to polynomial sum, and some choose to increase the numerator or decrease the denominator. The method adopted by many researchers is to select a fixed value, expressed as x/(ln (x) - a), where a>0 is a constant. For example, Legendre adopted a=1.08366 to correct it [3], but the effect is not ideal. In this paper, the calculation function of Mersenne's prime number [5] is adopted to dynamically modify the prime number theorem function x/ln (x), changing the traditional approach, and setting the constant a in the previous research as a function a=1+f (x), which is a small quantity that changes with x. Under this framework, the selection of f (x) is studied and compared. After a lot of experiments and further optimization, an expression of f (x) is determined, and the following definition is given.

Definition: p (x) is a revised prime number counting function composed of elementary functions:

$$p(x) = \frac{x}{\ln(x) - \left(1 + \frac{8\ln(2)}{15\sum_{y=1}^{x}\frac{1}{y^n}}\right)}$$

where: n=1.2 is the prime function constant.

## 2. Experimental verification

Use p (x) function to calculate the number of prime numbers that are less than the given order of magnitude, and compare with the calculation results of Li (x) , x/ln (x) and R (x) functions, see Table 1 to Table 3 and Figure 1 to Figure 6 for details. The experiments show that Li (x)>R (x)>x/ln (x); Similarly, Li (x)>p (x)>x/ln (x). In many cases, p (x) and R (x) are very similar to the prime number counting function π (x). With the increase of x, p (x), R (x) and π (x) cross each other, showing good approximation performance of p(x). The p (x) function has simple form, convenient calculation and relatively high accuracy.

Table 1 Calculation and comparison of the number of prime numbers less than the given magnitude

| x | π(x) | p(x) | x/ln(x) | Li(x) | p(x)/π(x) | x/ln(x) /π(x) | Li(x) /π(x) | (p(x)-π(x)) /π(x) | (x/ln(x)-π(x)) /π(x) | (Li(x)-π(x)) /π(x) |
|---|---|---|---|---|---|---|---|---|---|---|
| 100 | 25 | 29 | 22 | 30 | 1.1600 | 0.8800 | 1.2000 | 0.1600 | -0.1200 | 0.2000 |
| 1000 | 168 | 172 | 145 | 178 | 1.0238 | 0.8632 | 1.0595 | 0.0238 | -0.1369 | 0.0595 |
| 10000 | 1229 | 1230 | 1086 | 1246 | 1.0008 | 0.8836 | 1.0138 | 0.0008 | -0.1164 | 0.0138 |
| 20000 | 2262 | 2266 | 2019 | 2289 | 1.0018 | 0.8926 | 1.0120 | 0.0018 | -0.1074 | 0.0119 |
| 30000 | 3245 | 3249 | 2910 | 3277 | 1.0012 | 0.8968 | 1.0099 | 0.0012 | -0.1032 | 0.0099 |
| 40000 | 4203 | 4201 | 3375 | 4233 | 0.9995 | 0.8030 | 1.0071 | -0.0005 | -0.1970 | 0.0071 |
| 50000 | 5133 | 5130 | 4621 | 5167 | 0.9994 | 0.9003 | 1.0066 | -0.0006 | -0.0997 | 0.0066 |
| 1.0 10^5 | 9592 | 9578 | 8686 | 9630 | 0.9985 | 0.9055 | 1.0040 | -0.0015 | -0.0945 | 0.0040 |
| 5.0 10^5 | 41538 | 41488 | 38103 | 41606 | 0.9988 | 0.9173 | 1.0016 | -0.0012 | -0.0826 | 0.0016 |
| 1.0 10^6 | 78498 | 78459 | 72382 | 78628 | 0.9995 | 0.9221 | 1.0017 | -0.0005 | -0.0779 | 0.0017 |
| 2.0 10^6 | 148933 | 148819 | 137849 | 149059 | 0.9992 | 0.9256 | 1.0008 | -0.0008 | -0.0744 | 0.0008 |
| 1.0 10^7 | 664579 | 664472 | 620420 | 664918 | 0.9998 | 0.9336 | 1.0005 | -0.0002 | -0.0664 | 0.0005 |

Note: x is an integer; π (x) is the actual number function of prime numbers; p (x) is the revised prime counting function; x/ln (x) is the prime number function calculated by the prime number theorem; Li (x) is the prime number function calculated by the integration method of the prime number theorem.

Table 2 Calculation and comparison of the number of prime numbers within a given integer interval

| x | π(x) | p(x) | R(x) | p(x) /π(x)) | R(x) /π(x)) | (p(x)-π(x)) /π(x) | (R(x)-π(x)) /π(x) |
|---|---|---|---|---|---|---|---|
| [10000,10200] | 23 | 21.5365 | 21.5658 | 0.9365 | 0.9378 | -0.0635 | -0.0622 |
| [20000,20200] | 22 | 20.0734 | 20.1033 | 0.9123 | 0.9136 | -0.0877 | -0.0864 |
| [30000,30200] | 21 | 19.3023 | 19.3310 | 0.9190 | 0.9205 | -0.0810 | -0.0795 |
| [40000,40200] | 20 | 18.7889 | 18.8164 | 0.9395 | 0.9410 | -0.0605 | -0.0590 |
| [50000,50200] | 20 | 18.4086 | 18.4349 | 0.9205 | 0.9215 | -0.0795 | -0.0785 |
| [60000,60110] | 19 | 18.1088 | 18.1342 | 0.9532 | 0.9542 | -0.0468 | -0.0458 |
| [70000,70200] | 22 | 17.8627 | 17.8871 | 0.8118 | 0.8132 | -0.1882 | -0.1868 |
| [80000,80200] | 15 | 17.6547 | 17.6784 | 1.1767 | 1.1787 | 0.1767 | 0.1787 |
| [90000,90200] | 23 | 17.4752 | 17.4981 | 0.7600 | 0.7609 | -0.2400 | -0.2391 |
| [100000,100200] | 15 | 17.3175 | 17.3399 | 1.1547 | 1.1560 | 0.1547 | 0.1560 |
| [200000,200200] | 17 | 16.3465 | 16.3646 | 0.9618 | 0.9624 | -0.0382 | -0.0376 |
| [300000,300200] | 15 | 15.8267 | 15.8423 | 1.0553 | 1.0560 | 0.0553 | 0.0560 |
| [400000,400200] | 14 | 15.4773 | 15.4912 | 1.1057 | 1.1064 | 0.1057 | 0.1064 |
| [500000,500200] | 16 | 15.2166 | 15.2293 | 0.9513 | 0.9520 | -0.0488 | -0.0481 |
| [600000,600200] | 10 | 15.0100 | 15.0217 | 1.5010 | 1.5020 | 0.5010 | 0.5020 |
| [700000,700200] | 14 | 14.8395 | 14.8504 | 1.0592 | 1.0607 | 0.0593 | 0.0607 |
| [800000,800200] | 16 | 14.6951 | 14.7052 | 0.9188 | 0.9194 | -0.0813 | -0.0806 |
| [900000,900200] | 16 | 14.5698 | 14.5794 | 0.9106 | 0.9113 | -0.0894 | -0.0888 |
| [1000000,1000200] | 16 | 14.4596 | 14.4686 | 0.9100 | 0.9044 | -0.0900 | -0.0956 |
| Total deviation | | | | | | 2.2475 | 2.2479 |
| Average deviation | | | | | | 0.11829 | 0.11831 |
| Maximum deviation | | | | | | 0.5010 | 0.5020 |

Table 3 Calculation and comparison of the deviation of the prime number of p (x) and R (x) functions in a given interval

| Range x | p(x) | | | R(x) | | |
|---|---|---|---|---|---|---|
| | Maximum deviation Absolute value | Mean deviation Absolute value | Standard deviation | Maximum deviation Absolute value | Mean deviation Absolute value | Standard deviation |
| [10,10000] | 8.4782 | 3.2116 | 740.2433 | 5.5753 | 1.3185 | 739.2351 |
| [40000,50000] | 14.2954 | 3.5849 | 4671.3544 | 12.1111 | 3.7842 | 4673.6448 |

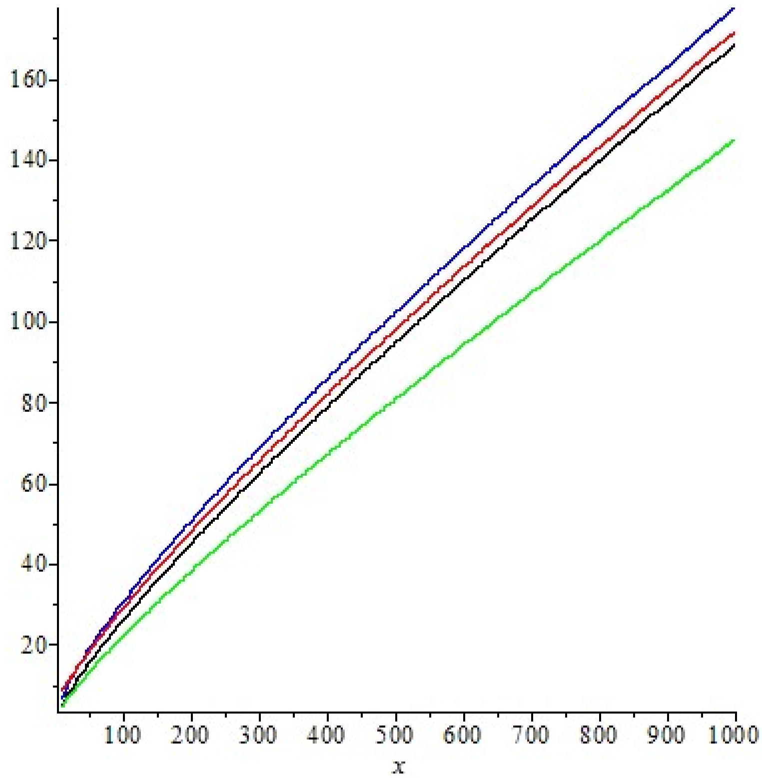


Figure 1 Blue Li (x), red p (x), black R (x), and green x/ln (x).

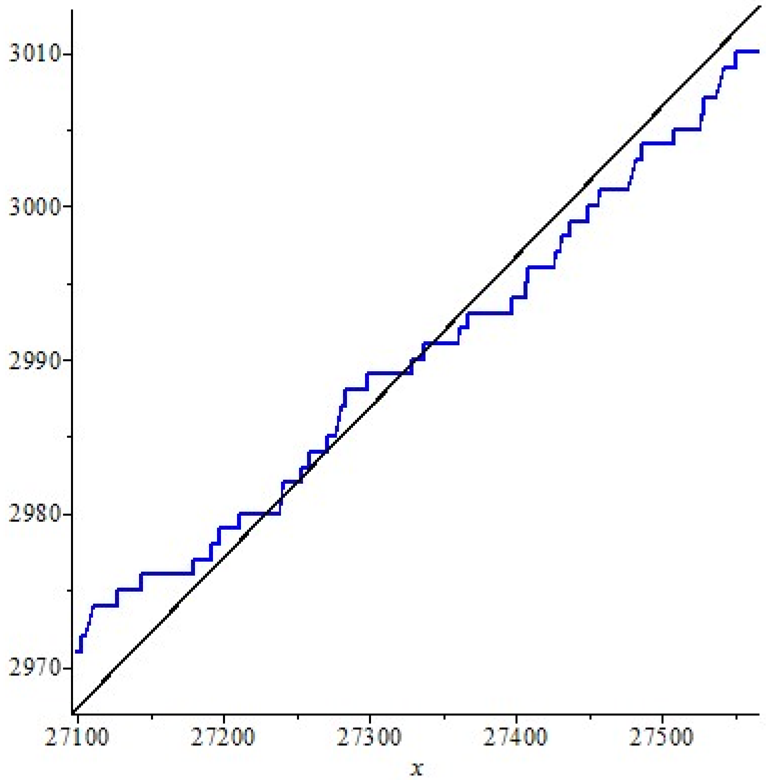


Figure 2 Blue π (x), red p (x) and black R (x), where p(x) is very closed to R(x), the same in the following.

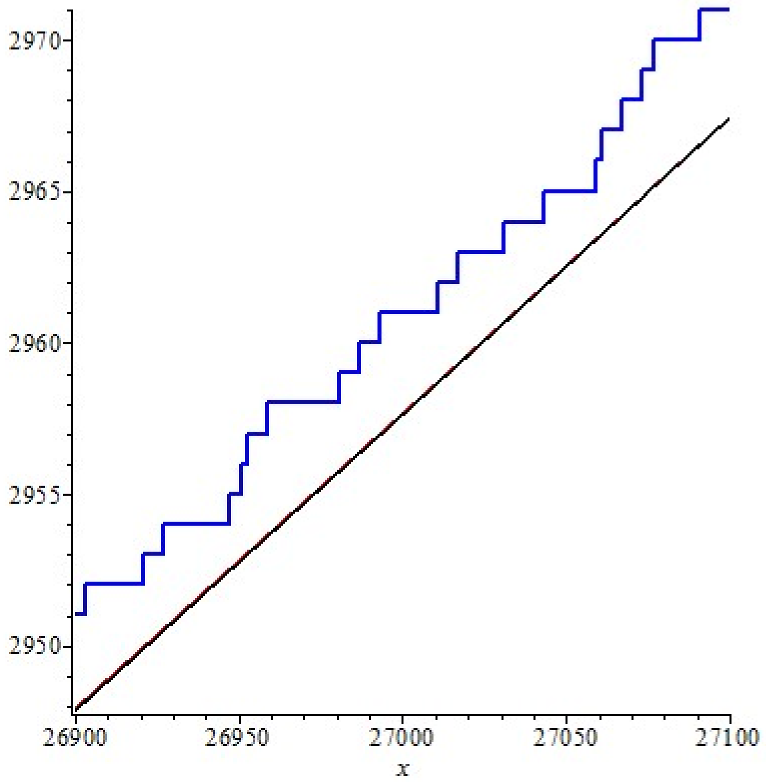


Figure 3 Blue π (x), red p (x) and black R (x).

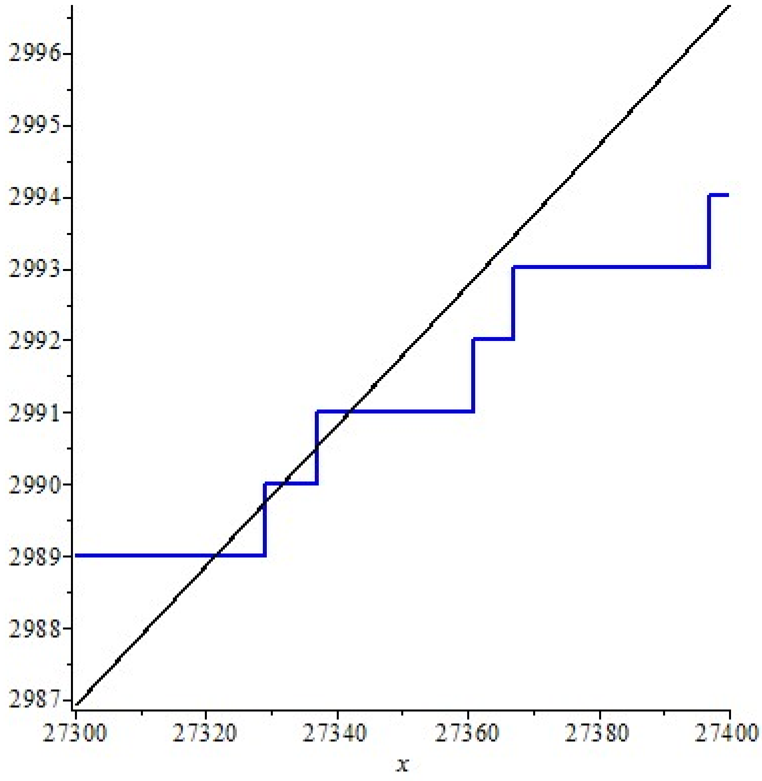


Figure 4 Blue π (x), red p (x) and black R (x).

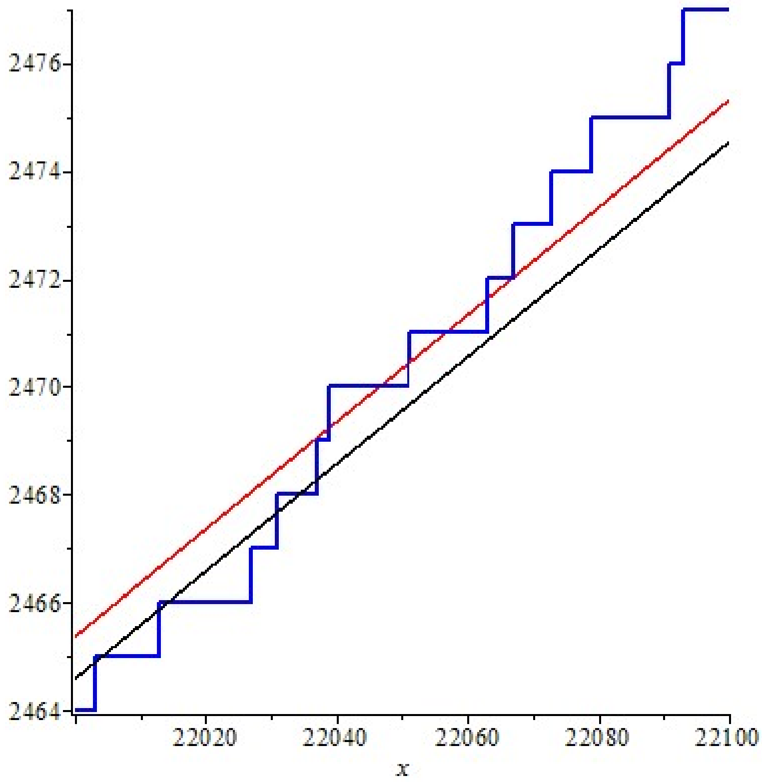


Figure 5 Blue π (x), red p (x) and black R (x).

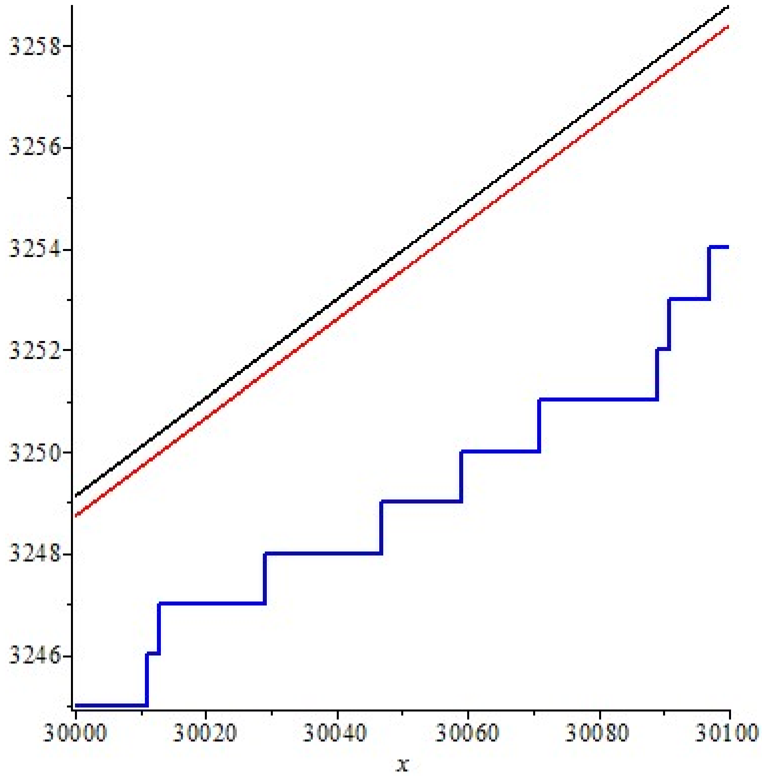


Figure 6 Blue π (x), red p (x) and black R (x).

3. Discussion and conclusion

Table 1 shows that the p (x) function is relatively accurate in calculating the number of prime numbers smaller than the given order of magnitude, which is significantly better than the calculation results of the prime number theorem function x/ln (x) and Li (x).

Table 2 shows that there are 19 groups of data for p (x) and R (x) in given intervals. Comparing the calculated values of p (x) and R (x) with those of π (x) , the variation direction of p (x) and R (x) is completely consistent, and the synchronization rate reaches 100.00%; There are 6 groups where the calculated values of p(x) and R(x) are greater than π (x), accounting for 31.58% and 13 groups less than π (x), 68.42% respectively; Among them, 7 groups of p (x) are superior to R (x) , accounting for 36.84%.

The calculated results of p (x) function are similar to those of R (x) function, but the maximum deviation and average deviation are slightly better than R (x) function.

Table 3 shows that the maximum deviation and average deviation of p (x) are slightly larger than R(x), but the variance is approximate, indicating that the data is reliable.

The results in Figures 1 to 6 show that the entanglement and crossing of p (x) and R (x) function curves and π (x) function curve often occur. It shows that the p (x) and R (x) are alternately optimal when calculating the number of prime numbers in different sections. In addition, the curves of p (x) , π (x) and R (x) are very close and fit well. It further shows that p (x) and R (x) are well representative, while x/ln (x) and Li (x) functions do not.

Because of the randomness of the distribution of prime numbers, the actual number of prime numbers always fluctuates above and below the value of the p (x) function, changing frequently and presenting a fluctuating state. Therefore, p (x) curve can be called prime number curve, and p (x) can be called prime number counting function.

It is easy to know that the calculation of p (x) is simple and the accuracy is higher than that of x/ln (x) and Li (x). With the increase of the given order of magnitude, the overall advantages of p (x) over x/ln (x) and Li (x) gradually expand. The p (x) expressed by elementary functions is an ideal prime number counting function, which has a wide application prospect.

## 4. References


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