Representations for $k\left(\sqrt{3}\,/\,2\right)$

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Abstract

In this note we give some integrals for $K\left(\sqrt{3}\,/\,2\right)$

Keywords: Elliptic integrals

I. Introduction

Recall that

$$\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{11}-...\tag{1}$$

and the complete elliptic integral of the first kind is defined by

$$K(k)=\int_0^{\pi/2}\frac{1}{\sqrt{1-k^2\sin^2 x}}dx=\int_0^1\frac{1}{\sqrt{\left(1-x^2\right)\left(1-k^2x^2\right)}}dx\ ,\ 0<k<1\tag{2}$$

in particular

$$K\left(\frac{\sqrt{3}}{2}\right)=\int_0^{\pi/2}\frac{1}{\sqrt{1-\frac{3}{4}\sin^2 x}}dx=\int_0^1\frac{1}{\sqrt{\left(1-x^2\right)\left(1-\frac{3}{4}x^2\right)}}dx\tag{3}$$

In this note we give some integrals for $k\left(\sqrt{3}\,/\,2\right)$.

II. Integrals

Entry 1.

$$K\left(\frac{\sqrt{3}}{2}\right)=\int_1^{\infty}\frac{1}{\sqrt{x\left(x-1\right)\left(x^2-x+1\right)}}dx\tag{4}$$

$$K\left(\frac{\sqrt{3}}{2}\right)=2\int_0^{\infty}\frac{1}{\sqrt{\left(1+x^2\right)\left(4+x^2\right)}}dx\tag{5}$$

$$K\left(\frac{\sqrt{3}}{2}\right)=\int_0^{1/2}\frac{\sqrt{x}+\sqrt{1-x}}{\sqrt{x\left(1-x\right)\left(1-x+x^2\right)}}dx\tag{6}$$

$$K\left(\frac{\sqrt{3}}{2}\right)=\int_0^{\infty}\frac{1}{\sqrt{x\,(1+x)\,(4+x)}}\,dx \tag{7}$$

$$K\left(\frac{\sqrt{3}}{2}\right)=2\int_0^{1}\frac{1}{\sqrt{1-x^2+x^4}}\,dx \tag{8}$$

$$K\left(\frac{\sqrt{3}}{2}\right)=\int_0^{1}\frac{1}{\sqrt{(1-x)\left(1-x+x^2\right)}}\,dx \tag{9}$$

$$K\left(\frac{\sqrt{3}}{2}\right)=\int_0^{1}\frac{1}{\sqrt{x\left(1-x+x^2\right)}}\,dx \tag{10}$$

Entry 2.

$$K\left(\frac{\sqrt{3}}{2}\right)=2\int_0^{\pi/2}\frac{1}{\sqrt{\sin^2 x+4\cos^2 x}}\,dx=2\int_0^{\pi/2}\frac{1}{\sqrt{1+3\cos^2 x}}\,dx=2\int_0^{\pi/2}\frac{1}{\sqrt{4-3\sin^2 x}}\,dx \tag{11}$$

Entry 3.

$$K\left(\frac{\sqrt{3}}{2}\right)=2\int_0^{1}\tanh^{-1}\left(\sqrt{\frac{2\left(1-x^2\right)}{2-x^2+x\sqrt{4-3\,x^2}}}\right)dx \tag{12}$$

Entry4.

$$K\left(\frac{\sqrt{3}}{2}\right)=\pi-\int_0^{1}\tan^{-1}\left(2\sqrt{\frac{x\,(2+x)}{(1-x)\,(3+x)}}\right)dx \tag{13}$$

$$K\left(\frac{\sqrt{3}}{2}\right)=\pi-\int_0^{1}\tan^{-1}\left(2\sqrt{\frac{(1-x)\,(3-x)}{x\,(4-x)}}\right)dx \tag{14}$$

Entry 5.

$$K\left(\frac{\sqrt{3}}{2}\right)=2\int_0^{\infty}\frac{1}{\sqrt{4+\sinh^2 x}}\,dx \tag{15}$$

$$K\left(\frac{\sqrt{3}}{2}\right)=2\int_0^{\infty}\frac{1}{\sqrt{3+\cosh^2 x}}\,dx \tag{16}$$

$$K\left(\frac{\sqrt{3}}{2}\right)=2\int_0^{\infty}\frac{1}{\sqrt{4\cosh^2 x-3\sinh^2 x}}\,dx \tag{17}$$

$$K\left(\frac{\sqrt{3}}{2}\right)=2\int_0^{\infty}\frac{1}{\sqrt{1+4\sinh^2 x}}\,dx \tag{18}$$

$$K\left(\frac{\sqrt{3}}{2}\right)=2\int_0^{\infty}\frac{1}{\sqrt{4\cosh^2 x-3}}\,dx \tag{19}$$

Entry 6.

$$K\left(\frac{\sqrt{3}}{2}\right) = \pi - 6\int_0^{\infty} \frac{x\tan^{-1}x}{\sqrt{1+x^2}\,\left(4+x^2\right)^{3/2}}\,dx \tag{20}$$

$$K\left(\frac{\sqrt{3}}{2}\right) = \pi - 6\int_0^{1} \frac{x\sin^{-1}x}{\left(4-3x^2\right)^{3/2}}\,dx \tag{21}$$

Entry 7.

$$K\left(\frac{\sqrt{3}}{2}\right) = 2\int_0^{1} \frac{1}{\sqrt{1-2x+5x^2-4x^3+x^4}}\,dx \tag{22}$$

Entry 8.

$$K\left(\frac{\sqrt{3}}{2}\right) = 2+\sqrt{2}\int_1^{2/\sqrt{3}} \left(\sqrt{1+\sqrt{4x^{-2}-3}} - \sqrt{1-\sqrt{4x^{-2}-3}}\right)dx \tag{23}$$

$$K\left(\frac{\sqrt{3}}{2}\right) = -2+\sqrt{2}\int_0^{1} \sqrt{1+\sqrt{4x^{-2}-3}}\,dx \tag{24}$$

Entry 9.

$$K\left(\frac{\sqrt{3}}{2}\right) = \int_0^{1} \tanh^{-1}\left(2\sqrt{\frac{1-x^2}{4-3x^2}}\right)dx \tag{25}$$

$$K\left(\frac{\sqrt{3}}{2}\right) = \int_0^{1} \sinh^{-1}\left(\frac{2\sqrt{1-x^2}}{x}\right)dx \tag{26}$$

$$K\left(\frac{\sqrt{3}}{2}\right) = \int_0^{1} \cosh^{-1}\left(\frac{\sqrt{4-3x^2}}{x}\right)dx \tag{27}$$

Entry 10.

$$K\left(\frac{\sqrt{3}}{2}\right) = 4\int_0^{1} \frac{1}{\sqrt{1+14x^2+x^4}}\,dx = 4\int_0^{\infty} \frac{e^{-x}}{\sqrt{1+14e^{-2x}+e^{-4x}}}\,dx \tag{28}$$

Entry 11.

$$K\left(\frac{\sqrt{3}}{2}\right) = 2\sqrt{2}\int_1^{\infty} \frac{1}{\sqrt{x\,(9+16x)\left(\sqrt{9+16x}-5\right)}}\,dx \tag{29}$$

$$K\left(\frac{\sqrt{3}}{2}\right) = 2\int_0^{1} \frac{1}{\sqrt{\left(1-2x+2x^2\right)\left(4-8x+5x^2\right)}}\,dx \tag{30}$$

Entry 12.

$$K\left(\frac{\sqrt{3}}{2}\right) = 1+\int_1^{\infty} f(x)\,dx \tag{31}$$

$$f(x) = \frac{1}{3} - \frac{2^{4/3}}{3}\left(\frac{27}{x^2} - 7 + \sqrt{32 + \left(\frac{27}{x^2} - 7\right)^2}\right)^{-1/3} + \frac{2^{-1/3}}{3}\left(\frac{27}{x^2} - 7 + \sqrt{32 + \left(\frac{27}{x^2} - 7\right)^2}\right)^{1/3} \tag{32}$$

Entry 13.

$$K\left(\frac{\sqrt{3}}{2}\right) = -4 + 4\int_0^1 \sqrt{f(x)}\, dx \tag{33}$$

$$f(x) = \frac{1}{2}\sqrt{\frac{2}{3} + g(x)} + \frac{1}{2}\sqrt{\frac{4}{3} - g(x) + \frac{2}{x^2\sqrt{\frac{2}{3} + g(x)}}} \tag{34}$$

$$g(x) = \frac{13\sqrt[3]{2}}{3}x^2(h(x))^{-1/3} + \frac{1}{3\sqrt[3]{2}\,x^2}(h(x))^{1/3} \tag{35}$$

$$h(x) = 27x^2 + 70x^6 + 3\sqrt{3}\,\sqrt{27x^4 + 140x^8 - 144x^{12}} \tag{36}$$

Entry 14.

$$K\left(\frac{\sqrt{3}}{2}\right) = \int_0^\infty \sqrt{\frac{1 + 2\cosh x}{1 + 2\cosh(2x)}}\, dx \tag{37}$$

$$K\left(\frac{\sqrt{3}}{2}\right) = 2\int_0^\infty \sqrt{\frac{\cosh x}{\cosh(3x)}}\, dx \tag{38}$$

Entry 15.

$$K\left(\frac{\sqrt{3}}{2}\right) = 2\int_0^1 \sqrt{\frac{1 + x^2 + x^4}{1 + x^4 + x^8}}\, dx = 2\int_1^\infty \sqrt{\frac{1 + x^2 + x^4}{1 + x^4 + x^8}}\, dx \tag{39}$$

$$K\left(\frac{\sqrt{3}}{2}\right) = \int_0^1 \sqrt{\frac{1 + x + x^2}{x\left(1 + x^2 + x^4\right)}}\, dx \tag{40}$$

$$K\left(\frac{\sqrt{3}}{2}\right) = 2\int_0^1 \sqrt{\frac{1 + x^2}{1 + x^6}}\, dx = 2\int_1^\infty \sqrt{\frac{1 + x^2}{1 + x^6}}\, dx \tag{41}$$

III. Endnote

Entry 16.

$$K\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi\sqrt{\pi}}{2\,(\Gamma(3/4))^2}F\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{2}, \frac{1}{4}\right) + \frac{\pi\sqrt{\pi}}{2\,(\Gamma(1/4))^2}F\left(\frac{3}{4}, \frac{3}{4}, \frac{3}{2}, \frac{1}{4}\right) \tag{42}$$

$$K\left(\frac{\sqrt{3}}{2}\right) = \frac{3\pi}{16}F\left(\frac{1}{2}, \frac{3}{2}, 2, \frac{3}{4}\right) + \frac{1}{2}\int_0^1 \sqrt{\frac{4 - 3x^2}{1 - x^2}}\, dx \tag{43}$$

$$K\left(\frac{\sqrt{3}}{2}\right) = 1 + \frac{3\pi}{16}F\left(\frac{1}{2}, \frac{3}{2}, 2, \frac{3}{4}\right) + \frac{1}{2}\int_2^\infty \left(1 - \sqrt{\frac{x^2 - 4}{x^2 - 3}}\right) dx \tag{44}$$

$$K\left(\frac{\sqrt{3}}{2}\right)=\frac{3\pi}{16}F\left(\frac{1}{2},\ \frac{3}{2},\ 2,\ \frac{3}{4}\right)+\int_0^1 \tanh^{-1}\left(2\sqrt{\frac{2\left(1-x^2\right)}{7+\sqrt{1+48x^2}}}\right)dx \tag{45}$$

$$K\left(\frac{\sqrt{3}}{2}\right)=\frac{3\pi}{16}F\left(\frac{1}{2},\ \frac{3}{2},\ 2,\ \frac{3}{4}\right)+2\int_0^\infty \sqrt{\frac{x^2+1}{\left(x^2+4\right)^3}}\ dx \tag{46}$$

$$K\left(\frac{\sqrt{3}}{2}\right)=\frac{3\pi}{16}F\left(\frac{1}{2},\ \frac{3}{2},\ 2,\ \frac{3}{4}\right)+\frac{1}{2}\int_0^\infty \sqrt{\frac{x^2+4}{\left(x^2+1\right)^3}}\ dx \tag{47}$$

Remark 1: $F(a,\ b,\ c,\ x)$ is the Gauss hypergeometric function.

Remark 2: $\Gamma(x)$ is the Gamma function.

References

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