ON SOLUTIONS TO ERDŐS-STRAUS EQUATION OVER CERTAIN INTEGER POWERS

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ABSTRACT. We apply the notion of the **olloid** to show that the Erdős-Straus equation

$$\frac{4}{n^{2^l}} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$$

has solutions for all $l \geq 1$ provided the equation

$$\frac{4}{n} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$$

has solution for a fixed $n > 4$.

1. Introduction

The Erdős-Straus equation is an equation of the form

$$\frac{4}{n} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}.$$

Paul Erdős and Ernst Straus conjectured in 1948 that the Erdős-Straus equation has solution in the positive integers for all $n \geq 2$. This is now known as the Erdős-Straus conjecture and still remains open. Nevertheless, substantial progress has been made in studying the conjecture accounting for a vast literature on the topic. It will be technically expedient to disprove the conjecture by finding a fraction of the form $\frac{4}{n}$ that has no three term representation. Computational search along this line has revealed that the conjecture is true for all positive integers $n \leq 10^{17}$ [1]. Effort in studying the conjecture has actually brought to bear various lines of attacks using certain modular and polynomial identities of the fraction $\frac{4}{n}$ for all $n \in \mathbb{N}$ with $n \geq 2$. Another noteworthy approach is to study the conjecture using the tools of Diophantine analysis, since the Erdős-Straus equation can be recast in the form of the Diophantine equation $4xyz = n(xy+yz+xz)$ [5]. The conjecture is much more tractable along the realms of modular arithmetic; indeed, the conjecture is solvable modulo prime powers but there appears to be no clear path to piecing these together to yield a solution to the conjecture. The truth of the conjecture holds for general congruence classes. In particular, the expansion

$$\frac{4}{n} = \frac{1}{n} + \frac{1}{\frac{n+1}{3}} + \frac{1}{\frac{n(n+1)}{3}}$$

holds in $2 \pmod 3$. A general polynomial identity devised by Mordell is also used to provide the unit fraction expansion for $\frac{4}{n}$ in $2 \pmod 3$ and above, $3 \pmod 4$, 2 or $3 \pmod 5$, $3, 5, 6 \pmod 7$ or $5 \pmod 8$. With an ongoing search for a solution or

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a counter-example, it had been shown that the natural density of counter-examples - if any - to the Erdős-Straus is zero [3]. The use of modular identities have always been useful in proving the conjecture in various congruence classes but it certainly has it's limitations. It is known that if $n \equiv r \pmod p$ for any prime number p and r is not congruent to a quadratic residue then the exist a modular identity for $\frac{4}{n}$ and, hence, a three term unit expansion for $\frac{4}{n}$ in the congruence class $r \pmod p$ exists [4]. Since 1 is a square, it follows that there is no polynomial identity for $\frac{4}{n}$ for $n \equiv 1 \pmod 3$ so that there is no complete covering system of modular identities. The Erdős-Straus conjecture could easily be proved if there exist a unit fraction expansion for $\frac{4}{n}$ in distinct moduli forming a complete covering system, but this approach seems impossible. The number of solutions to the Erdős-Straus equation has also been studied and various upper bound for their solution is now known. In fact, Elsholtz and Tao showed that the average number of solutions to the Erdős-Straus equation $\frac{4}{n} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$ over the primes grows polylogarithmically in n [2]. In this paper we show that we can avoid the solutions of the Erdős-Straus equation $\frac{4}{n} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$ in certain arithmetic progression. In particular, we obtain the result

Theorem 1.1. *Let $\sum\limits_{k}$ denotes a sum of exactly k terms. If the equation*

$$\sum_{3} \frac{1}{s} = \frac{4}{n}$$

has a solution for fixed $n > 4$, then the equation

$$\sum_{3} \frac{1}{s} = \frac{4}{n^{2^l}}$$

also has a solution for all $l \geq 1$

This result is a consequence of the more fundamental result using the notion of the **olloid**.

Lemma 1.2 (Expansion principle)**.** *Let $\mathbb{F}_s^k$ be an s-dimensional **olloid** of degree k for a fixed $k \in \mathbb{N}$ with $k > 1$. If $g : \mathbb{N} \longrightarrow \mathbb{R}^+$ is a generator with continuous derivative on $[1, s]$ and decreasing on $\mathbb{R}^+$ such that*

$$1 - \frac{1}{g(s)^r} > \int_1^s \frac{g'(t)}{g(t)^2} dt + \frac{1}{g(s)} \int_1^s \frac{g'(t)}{g(t)^2} dt + \cdots + \frac{1}{g(s)^{r-1}} \int_1^s \frac{g'(t)}{g(t)^2} dt$$

*for $r \in \mathbb{N}$ then $g : \mathbb{N} \longrightarrow \mathbb{R}^+$ is also a generator of the **olloid** $\mathbb{F}_s^{k+r}$ of degree $k + r$.*

2. The notion of the olloid

In this section we launch the notion of the **olloid** and prove a fundamental lemma, which will be relevant for our studies in the sequel.

Definition 2.1. Let $\mathbb{F}_s^k := \left\{ (u_1, u_2, \ldots, u_s) \in \mathbb{R}^s \mid \sum\limits_{i=1}^{s} u_i^k = 1, \ k \geq 1 \right\}$. Then we call $\mathbb{F}_s^k$ an s-dimensional **olloid** of degree $k \geq 1$. We say $g : \mathbb{N} \longrightarrow \mathbb{R}$ is a generator of the s-dimensional olloid of degree k if there exists some vector $(v_1, v_2, \ldots, v_s) \in \mathbb{F}_s^k$ such that $v_i = g(i)$ for each $1 \leq i \leq s$.

Question 2.2. Does there exists a fixed generator $g:\mathbb{N}\longrightarrow\mathbb{R}$ with infinitely many olloids?

Remark 2.3. While it may be difficult to provide a general answer to question 2.2, we can in fact provide an answer by imposition certain conditions for which the generator of the **olloid** must satisfy. In particular, we launch a basic and a fundamental principle relevant for our studies in the sequel.

Lemma 2.4 (Expansion principle)**.** *Let $\mathbb{F}_s^k$ be an s-dimensional **olloid** of degree $k\geq 1$ for a fixed $k\in\mathbb{N}$. If $g:\mathbb{N}\longrightarrow\mathbb{R}^+$ is a generator with continuous derivative on $[1,s]$ and decreasing on $\mathbb{R}^+$ such that*

$$1-\frac{1}{g(s)^r}>\int\limits_{1}^{s}\frac{g'(t)}{g(t)^2}dt+\frac{1}{g(s)}\int\limits_{1}^{s}\frac{g'(t)}{g(t)^2}dt+\cdots+\frac{1}{g(s)^{r-1}}\int\limits_{1}^{s}\frac{g'(t)}{g(t)^2}dt$$

*for $r\in\mathbb{N}$ then $g:\mathbb{N}\longrightarrow\mathbb{R}^+$ is also a generator of the **olloid** $\mathbb{F}_s^{k+r}$ of degree $k+r$.*

Proof. Suppose $g:\mathbb{N}\longrightarrow\mathbb{R}^+$ is a generator of the **olloid** $\mathbb{F}_s^k$ with continuous derivative on $[1,s]$. Then there exists a vector $(v_1,v_2,\ldots,v_s)\in\mathbb{F}_s^k$ such that $v_i=g(i)$ for each $1\leq i\leq s$, so that we can write

$$\sum_{i=1}^{s}g(i)^k=1.$$

Let us assume to the contrary that there exists no $r\in\mathbb{N}$ such that $g:\mathbb{N}\longrightarrow\mathbb{R}^+$ is a generator of the **olloid** $\mathbb{F}_s^{k+r}$. By applying the summation by parts, we obtain the inequality

$$\frac{1}{g(s)}\sum_{i=1}^{s}g(i)^{k+1}\geq 1-\int\limits_{1}^{s}\frac{g'(t)}{g(t)^2}dt \tag{2.1}$$

by using the inequality

$$\sum_{i=1}^{s}g(i)^{k+1}<\sum_{i=1}^{s}g(i)^k=1.$$

By applying summation by parts on the left side of (2.1) and using the contrary assumption, we obtain further the inequality

$$\frac{1}{g(s)^2}\sum_{i=1}^{s}g(i)^{k+2}\geq 1-\int\limits_{1}^{s}\frac{g'(t)}{g(t)^2}dt-\frac{1}{g(s)}\int\limits_{1}^{s}\frac{g'(t)}{g(t)^2}dt. \tag{2.2}$$

By induction we can write the inequality as

$$\frac{1}{g(s)^r}\sum_{i=1}^{s}g(i)^{k+r}\geq 1-\int\limits_{1}^{s}\frac{g'(t)}{g(t)^2}dt-\frac{1}{g(s)}\int\limits_{1}^{s}\frac{g'(t)}{g(t)^2}dt-\cdots-\frac{1}{g(s)^{r-1}}\int\limits_{1}^{s}\frac{g'(t)}{g(t)^2}dt$$

for any $r\geq 2$ with $r\in\mathbb{N}$. Since $g:\mathbb{N}\longrightarrow\mathbb{R}^+$ is decreasing, it follows that

$$1-\int\limits_{1}^{s}\frac{g'(t)}{g(t)^2}dt-\frac{1}{g(s)}\int\limits_{1}^{s}\frac{g'(t)}{g(t)^2}dt-\cdots-\frac{1}{g(s)^{r-1}}\int\limits_{1}^{s}\frac{g'(t)}{g(t)^2}dt>1$$

and using the requirement

$$1-\frac{1}{g(s)^r} > \int\limits_{1}^{s}\frac{g'(t)}{g(t)^2}dt + \frac{1}{g(s)}\int\limits_{1}^{s}\frac{g'(t)}{g(t)^2}dt + \cdots + \frac{1}{g(s)^{r-1}}\int\limits_{1}^{s}\frac{g'(t)}{g(t)^2}dt$$

for $r \in \mathbb{N}$, we have the inequality

$$\begin{aligned}1 &= \sum_{i=1}^{s} g(i)^k \\ &\geq \sum_{i=1}^{s} g(i)^{k+r} > 1\end{aligned}$$

which is absurd. This completes the proof of the Lemma. $\square$

Lemma 2.4 - albeit fundamental - is ultimately useful for our study of variants and possibly extensions of the Erdős-Straus equation. It can be seen as a tool for extending the solution of equations of the form

$$\sum_{i=1}^{s} g(i)^k = 1$$

for $k > 1$ - under the presumption that it exists - to the solution of equations of the form

$$\sum_{i=1}^{s} g(i)^{k+r} = 1$$

for a fixed $r \in \mathbb{N}$ under some special requirements of the generator $g : \mathbb{N} \longrightarrow \mathbb{R}$.

3. Application to solutions of the Erdős-Straus equation

In this section we apply the notion of the **olloid** to study the non-existence of solutions of the Erdős-Straus equation over certain powers of integers n^{2^k} for all $k \geq 1$. This is an outgrowth of Lemma 2.4.

Theorem 3.1. *Let* $\sum\limits_{k}$ *denotes a sum of exactly* k *terms. If the equation*

$$\sum_{3}\frac{1}{s} = \frac{4}{n}$$

has a solution for fixed $n > 4$*, then the equation*

$$\sum_{3}\frac{1}{s} = \frac{4}{n^{2^l}}$$

also has a solution for all $l \geq 1$

Proof. Let us assume that the equation

$$\sum_{3}\frac{1}{s} = \frac{4}{n}$$

has a solution for a fixed $n > 4$. The equation above can be recast in the form

$$\sum_{3}\frac{n}{4s} = 1$$

for a fixed $n > 4$. Let us take the generator $g(s) = \frac{n}{4s}$ for a fixed $n > 4$, then we note that $g(s)$ is decreasing on $[1, u]$ and has continuous derivative on $[1, u]$ for all $u > 1$. We note that

$$1 - \frac{1}{g(u)} = 1 - \frac{4u}{n}$$

and

$$\int_1^u \frac{g'(t)}{g(t)^2} dt = \frac{4}{n} - \frac{4u}{n}$$

so that if

$$1 - \frac{4u}{n} \leq \frac{4}{n} - \frac{4u}{n}$$

with the choice $r = 1$ in the proof of Lemma 2.4 then $n \leq 4$, violating the inequality $n > 4$. Appealing to Lemma 2.4 then the equation

$$\sum_3 \frac{n^2}{16s^2} = 1$$

also has a solution, which can further be written in the form

$$\sum_3 \frac{1}{m} = \frac{4}{n^2} \tag{3.1}$$

by making the substitution $m = 4s^2$. By iterating the argument on (3.1) in the sense of Lemma 2.4, it must be that the equation

$$\sum_3 \frac{1}{t} = \frac{4}{n^4}$$

also has a solution. The claim follows by iterating repeatedly in this manner. □

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