A COMPLETE PROOF OF THE CONJECTURE $c < rad^{1.63}(abc)$

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To the memory of my Father who taught me arithmetic,
To my wife Wahida, my daughter Sinda and my son Mohamed Mazen
To Prof. A. Nitaj for his work on the abc conjecture

Abstract. In this paper, we consider the *abc* conjecture, we will give the proof that the conjecture $c < rad^{1.63}(abc)$ is true. It constitutes the key to resolve the *abc* conjecture.

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1. Introduction and notations

Let a be a positive integer, $a = \prod_i a_i^{\alpha_i}$, a_i prime integers and $\alpha_i \geq 1$ positive integers. We call *radical* of a the integer $\prod_i a_i$ noted by $rad(a)$. Then a is written as:

$$a = \prod_i a_i^{\alpha_i} = rad(a).\prod_i a_i^{\alpha_i - 1} \tag{1}$$

We denote:

$$\mu_a = \prod_i a_i^{\alpha_i - 1} \Longrightarrow a = \mu_a.rad(a) \tag{2}$$

The *abc* conjecture was proposed independently in 1985 by David Masser of the University of Basel and Joseph Œsterlé of Pierre et Marie Curie University (Paris 6) [1]. It describes the distribution of the prime factors of two integers with those of its sum. The definition of the *abc* conjecture is given below:

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Conjecture 1.1. *(abc Conjecture): For each $\epsilon > 0$, there exists $K(\epsilon)$ such that if a, b, c positive integers relatively prime with $c = a + b$, then :*

$$c < K(\epsilon).rad^{1+\epsilon}(abc) \tag{3}$$

where K is a constant depending only of ϵ.

We know that numerically, $\dfrac{Logc}{Log(rad(abc))} \leq 1.629912$ [2]. It concerned the best example given by E. Reyssat [2]:

$$2 + 3^{10}.109 = 23^5 \Longrightarrow c < rad^{1.629912}(abc) \tag{4}$$

A conjecture was proposed that $c < rad^2(abc)$ [3]. In 2012, A. Nitaj [4] proposed the following conjecture:

Conjecture 1.2. *Let a, b, c be positive integers relatively prime with $c = a + b$, then:*

$$c < rad^{1.63}(abc) \tag{5}$$

$$abc < rad^{4.42}(abc) \tag{6}$$

In this paper, we will give the proof of the conjecture given by (5) that constitutes the key to obtain the proof of the *abc* conjecture using classical methods with the help of some theorems from the field of the number theory.

2. The Proof of the conjecture $c < rad^{1.63}(abc)$

Let a, b, c be positive integers, relatively prime, with $c = a + b$, $b < a$ and $R = rad(abc)$, $c = \prod_{j'=1}^{j'=J'} c_{j'}^{\beta_{j'}}, \beta_{j'} \geq 1$, $c_{j'} \geq 2$ prime integers.

In the following, we will give the proof of the conjecture $c < rad^{1.63}(abc)$.

Proof. :

2.1. **Trivial cases:** - We suppose that $c < rad(abc)$, then we obtain:

$$c < rad(abc) < rad^{1.63}(abc) \Longrightarrow \boxed{c < R^{1.63}}$$

and the condition (5) is satisfied.

- We suppose that $c = rad(abc)$, then a, b, c are not coprime, case to reject.

In the following, we suppose that $c > rad(abc)$ and a, b and c are not all prime numbers.

- We suppose $\mu_a \leq rad^{0.63}(a)$. We obtain :

$$c = a + b < 2a \leq 2rad^{1.63}(a) < rad^{1.63}(abc) \Longrightarrow c < rad^{1.63}(abc) \Longrightarrow \boxed{c < R^{1.63}}$$

Then (5) is satisfied.

- We suppose $\mu_c \leq rad^{0.63}(c)$. We obtain :

$$c = \mu_c rad(c) \leq rad^{1.63}(c) < rad^{1.63}(abc) \Longrightarrow \boxed{c < R^{1.63}}$$

and the condition (5) is satisfied.

2.2. **We suppose $\mu_c > rad^{0.63}(c)$ and $\mu_a > rad^{0.63}(a)$.**

2.2.1. *Case : $rad^{0.63}(c) < \mu_c \le rad^{1.63}(c)$ and $rad^{0.63}(a) < \mu_a \le rad^{1.63}(a)$.*
We can write:

$$\left.\begin{array}{l} \mu_c \le rad^{1.63}(c) \Longrightarrow c \le rad^{2.63}(c) \\ \\ \mu_a \le rad^{1.63}(a) \Longrightarrow a \le rad^{2.63}(a) \end{array}\right\} \Longrightarrow ac \le rad^{2.63}(ac) \Longrightarrow a^2 < ac \le rad^{2.63}(ac)$$

$$\Longrightarrow a < rad^{1.315}(ac) \Longrightarrow c < 2a < 2rad^{1.315}(ac) < rad^{1.63}(abc)$$

$$\Longrightarrow \boxed{c = a + b < R^{1.63}}$$

2.2.2. *Case : $rad^{1.63}(c) < \mu_c$ or $rad^{1.63}(a) < \mu_a$.* **I -** We suppose that $rad^{1.63}(c) < \mu_c$ and $rad^{1.63}(a) < \mu_a \le rad^2(a)$:

I-1- Case $rad(a) < rad(c)$:
In this case $a = \mu_a.rad(a) \le rad^3(a) \le rad^{1.63}(a)rad^{1.37}(a) < rad^{1.63}(a).rad^{1.37}(c)$ $\Longrightarrow c < 2a < 2rad^{1.63}(a).rad^{1.37}(c) < rad^{1.63}(abc) \Longrightarrow \boxed{c < R^{1.63}}$.

I-2- Case $rad(c) < rad(a) < rad^{\frac{1.63}{1.37}}(c)$: As $a \le rad^{1.63}(a).rad^{1.37}(a) < rad^{1.63}(a).rad^{1.63}(c) \Longrightarrow c < 2a < 2rad^{1.63}(a).rad^{1.63}(c) < R^{1.63} \Longrightarrow \boxed{c < R^{1.63}}$.

I-3- Case $rad^{\frac{1.63}{1.37}}(c) < rad(a)$:

I-3-1- We suppose $rad^{1.63}(c) < \mu_c \le rad^{2.26}(c)$, we obtain:

$$c \le rad^{3.26}(c) \Longrightarrow c \le rad^{1.63}(c).rad^{1.63}(c) \Longrightarrow$$

$$c < rad^{1.63}(c).rad^{1.37}(a) < rad^{1.63}(c).rad^{1.63}(a).rad^{1.63}(b) = R^{1.63} \Longrightarrow \boxed{c < R^{1.63}}$$

I-3-2- We suppose $\mu_c > rad^{2.26}(c) \Longrightarrow c > rad^{3.26}(c)$.

I-3-2-1- We consider the case $\mu_a = rad^2(a) \Longrightarrow a = rad^3(a)$ and $c = a + 1$. Then, we obtain that $X = rad(a)$ is a solution in positive integers of the equation:

$$X^3 + 1 = c \tag{7}$$

I-3-2-1-1- We suppose that $c = rad^n(c)$ with $n \ge 4$, we obtain the equation:

$$rad^n(c) - rad^3(a) = 1 \tag{8}$$

But the solutions of the equation (8) are [5] :($rad(c) = 3, n = 2, rad(a) = +2$), it follows the contradiction with $n \ge 4$ and the case $c = rad^n(c), n \ge 4$ is to reject.

I-3-2-1-2- In the following, we will study the cases $\mu_c = A.rad^n(c)$ with $rad(c) \nmid A, n \ge 0$. The above equation (7) can be written as :

$$(X + 1)(X^2 - X + 1) = c \tag{9}$$

Let δ one divisor of c so that :

$$X + 1 = \delta \tag{10}$$

$$X^2 - X + 1 = \frac{c}{\delta} = m = \delta^2 - 3X \tag{11}$$

We recall that $rad(a) > rad^{\frac{1.63}{1.37}}(c)$.

I-3-2-1-2-1- We suppose $\delta = l.rad(c)$. We have $\delta = l.rad(c) < c = \mu_c.rad(c) \Longrightarrow l < \mu_c$. As $\dfrac{c}{\delta} = \dfrac{\mu_c rad(c)}{l rad(c)} = \dfrac{\mu_c}{l} = m = \delta^2 - 3X \Longrightarrow \mu_c = l.m = l(\delta^2 - 3X)$. From $m = \delta^2 - 3X)$ and $X = rad(a)$, we obtain:

$$m = l^2 rad^2(c) - 3rad(a) \Longrightarrow 3rad(a) = l^2 rad^2(c) - m$$

A- Case $3|m \Longrightarrow m = 3m'$, $m' > 1$: As $\mu_c = ml = 3m'l \Longrightarrow 3|rad(c)$ and $(rad(c), m')$ not coprime. We obtain:

$$rad(a) = l^2 rad(c).\frac{rad(c)}{3} - m'$$

It follows that a, c are not coprime, then the contradiction.

B - Case $m = 3 \Longrightarrow \mu_c = 3l \Longrightarrow c = 3lrad(c) = 3\delta = \delta(\delta^2 - 3X) \Longrightarrow \delta^2 = 3(1 + X) = 3\delta \Longrightarrow \delta = lrad(c) = 3 \Longrightarrow c = 3\delta = 9 = a + 1 \Longrightarrow a = 8 \Longrightarrow c = 9 < (2 \times 3)^{1.63} \approx 18.55$, it is a trivial case and the conjecture is true.

I-3-2-1-2-2- We suppose $\delta = l.rad^2(c), l \geq 2$. If $n = 0$ then $\mu_c = A$ and from the equation above (11):

$$m = \frac{c}{\delta} = \frac{\mu_c.rad(c)}{lrad^2(c)} = \frac{A.rad(c)}{lrad^2(c)} = \frac{A}{lrad(c)} \Rightarrow rad(c)|A$$

It follows the contradiction with the hypothesis above $rad(c) \nmid A$.

I-3-2-1-2-3- We suppose $\delta = lrad^2(c), l \geq 2$ and in the following $n > 0$. As $m = \dfrac{c}{\delta} = \dfrac{\mu_c.rad(c)}{lrad^2(c)} = \dfrac{\mu_c}{lrad(c)}$, if $lrad(c) \nmid \mu_c$ then the case is to reject. We suppose $lrad(c)|\mu_c \Longrightarrow \mu_c = m.lrad(c)$, with $m, rad(c)$ not coprime, then $\dfrac{c}{\delta} = m = \delta^2 - 3rad(a)$.

C - Case $m = 1 = c/\delta \Longrightarrow \delta^2 - 3rad(a) = 1 \Longrightarrow (\delta - 1)(\delta + 1) = 3rad(a) = rad(a)(\delta + 1) \Longrightarrow \delta = 2 = l.rad^2(c)$, then the contradiction.

D - Case $m = 3$, we obtain $3(1 + rad(a)) = \delta^2 = 3\delta \Longrightarrow \delta = 3 = lrad^2(c)$. Then the contradiction.

E - Case $m \neq 1, 3$, we obtain: $3rad(a) = l^2 rad^4(c) - m \Longrightarrow rad(a)$ and $rad(c)$ are not coprime. Then the contradiction.

I-3-2-1-2-4- We suppose $\delta = l.rad^n(c), l \geq 2$ with $n \geq 3$. $c = \mu_c.rad(c) = lrad^n(c)(\delta^2 - 3rad(a))$ and $m = \delta^2 - 3rad(a) = \delta^2 - 3X$.

F - As seen above (paragraphs C,D), the cases $m = 1$ and $m = 3$ give contradictions, it follows the reject of these cases.

G - Case $m \neq 1, 3$. Let q be a prime that divides m (q can be equal to m), it follows $q|(\mu_c = l.m) \Longrightarrow q = c_{j'_0} \Longrightarrow c_{j'_0}|\delta^2 \Longrightarrow c_{j'_0}|3rad(a)$. Then $rad(a)$

and $rad(c)$ are not coprime. It follows the contradiction.

I-3-2-1-2-5- We suppose $\delta = \prod_{j\in J_1} c_j^{\beta_j}$, $\beta_j \geq 1$ with at least one $j_0 \in J_1$ with:

$$\beta_{j_0} \geq 2, \quad rad(c) \nmid \delta \tag{12}$$

We can write:

$$\delta = \mu_\delta.rad(\delta), \quad rad(c) = r.rad(\delta), \quad r > 1, \quad (r, \mu_\delta) = 1 \tag{13}$$

Then, we obtain:

$$c = \mu_c.rad(c) = \mu_c.r.rad(\delta) = \delta(\delta^2 - 3X) = \mu_\delta.rad(\delta)(\delta^2 - 3X) \Longrightarrow$$
$$r.\mu_c = \mu_\delta(\delta^2 - 3X) \tag{14}$$

- We suppose $\mu_c = \mu_\delta \Longrightarrow r = \delta^2 - 3X = (\mu_c.rad(\delta))^2 - 3X$. As $\delta < \delta^2 - 3X \Longrightarrow r > \delta \Longrightarrow rad(c) > r > (\mu_c.rad(\delta) = A.rad^n(c)rad(\delta)) \Longrightarrow 1 > A.rad^{n-1}(\delta)$, then the contradiction.

- We suppose $\mu_c < \mu_\delta$. As $rad(a) = \delta - 1 = \mu_\delta rad(\delta) - 1$, we obtain:

$$rad(a) > \mu_c.rad(\delta) - 1 > 0 \Longrightarrow rad(ac) > c.rad(\delta) - rad(c) > 0$$

As $c = 1 + a$ and we consider the cases $c > rad(ac)$, then:

$$c > rad(ac) > c.rad(\delta) - rad(c) > 0 \Longrightarrow c > c.rad(\delta) - rad(c) > 0 \Longrightarrow$$
$$1 > rad(\delta) - \frac{rad(c)}{c} > 0, \quad rad(\delta) \geq 2 \Longrightarrow \text{The contradiction} \tag{15}$$

- We suppose $\mu_c > \mu_\delta$. In this case, from the equation (14) and as $(r, \mu_\delta) = 1$, it follows we can write:

$$\mu_c = \mu_1.\mu_2, \quad \mu_1, \mu_2 > 1,$$
$$c = \mu_c rad(c) = \mu_1.\mu_2.rad(\delta).r = \delta.(\delta^2 - 3X),$$

We do a choice so that $\quad \mu_2 = \mu_\delta, \quad r.\mu_1 = \delta^2 - 3X \Longrightarrow \delta = \mu_2.rad(\delta)$.

** *1-* We suppose $(\mu_1, \mu_2) \neq 1$, then $\exists\, c_{j_0}$ so that $c_{j_0}|\mu_1$ and $c_{j_0}|\mu_2$. But $\mu_\delta = \mu_2 \Rightarrow c_{j_0}^2|\delta$. From $3X = \delta^2 - r\mu_1 \Longrightarrow c_{j_0}|3X \Longrightarrow c_{j_0}|X$ or $c_{j_0} = 3$.

- If $c_{j_0}|(X = rad(a))$, it follows the contradiction with $(c, a) = 1$.
- If $c_{j_0} = 3$. We have $r\mu_1 = \delta^2 - 3X = \delta^2 - 3(\delta - 1) \Longrightarrow \delta^2 - 3\delta + 3 - r.\mu_1 = 0$.

As $3|\mu_1 \Longrightarrow \mu_1 = 3^k\mu'_1, 3 \nmid \mu'_1, k \geq 1$, we obtain:

$$\delta^2 - 3\delta + 3(1 - 3^{k-1}r\mu'_1) = 0 \tag{16}$$

** *1-1-* We consider the case $k > 1 \Longrightarrow 3 \nmid (1 - 3^{k-1}r\mu'_1)$. Let us recall the Eisenstein criterion [6]:

Theorem 2.1. ***(Eisenstein Criterion)*** *Let $f = a_0 + \cdots + a_nX^n$ be a polynomial $\in \mathbb{Z}[X]$. We suppose that $\exists p$ a prime number so that $p \nmid a_n$, $p|a_i$, $(0 \leq i \leq n-1)$, and $p^2 \nmid a_0$, then f is irreducible in $\mathbb{Q}$.*

We apply Eisenstein criterion to the polynomial $R(Z)$ given by:

$$R(Z) = Z^2 - 3Z + 3(1 - 3^{k-1}r\mu'_1) \tag{17}$$

then:

- $3 \nmid 1$, - $3|(-3)$,- $3|3(1 - 3^{k-1}r\mu'_1)$, and - $3^2 \nmid 3(1 - 3^{k-1}r\mu'_1)$.

It follows that the polynomial $R(Z)$ is irreducible in $\mathbb{Q}$, then, the contradiction with $R(\delta)=0$.

** *1-2-* We consider the case $k=1$, then $\mu_1=3\mu'_1$ and $(\mu'_1,3)=1$, we obtain:

$$\delta^2-3\delta+3(1-r\mu'_1)=0 \tag{18}$$

** *1-2-1-* We consider that $3\nmid(1-r.\mu'_1)$, we apply the same Eisenstein criterion to the polynomial $R'(Z)$ given by:

$$R'(Z)=Z^2-3Z+3(1-r\mu'_1)$$

and we find a contradiction with $R'(\delta)=0$.

** *1-2-2-* We consider that:

$$3|(1-r.\mu'_1)\Longrightarrow r\mu'_1-1=3^i.h,\ i\geq 1,\ 3\nmid h, h\in\mathbb{N}^* \tag{19}$$

δ is an integer root of the polynomial $R'(Z)$:

$$R'(Z)=Z^2-3Z+3(1-r\mu'_1)=0 \tag{20}$$

The discriminant of $R'(Z)$ is:

$$\Delta=3^2+3^{i+1}\times 4.h$$

As the root δ is an integer, it follows that $\Delta=t^2>0$ with t a positive integer. We obtain:

$$\Delta=3^2(1+3^{i-1}\times 4h)=t^2 \tag{21}$$

$$\Longrightarrow 1+3^{i-1}\times 4h=q^2>1, q\in\mathbb{N}^* \tag{22}$$

We can write the equation (18) as :

$$\delta(\delta-3)=3^{i+1}.h\Longrightarrow 3^3\mu'_1\frac{rad(\delta)}{3}.\big(\mu'_1rad(\delta)-1\big)=3^{i+1}.h\Longrightarrow \tag{23}$$

$$\mu'_1\frac{rad(\delta)}{3}.\big(\mu'_1rad(\delta)-1\big)=h \tag{24}$$

We obtain $i=2$ and $q^2=1+12h=1+4\mu'_1rad(\delta)(\mu'_1rad(\delta)-1)$. Then, q satisfies :

$$q^2-1=12h=4\mu'_1rad(\delta)(\mu'_1rad(\delta)-1)\Longrightarrow \tag{25}$$

$$\tfrac{(q-1)}{2}.\tfrac{(q+1)}{2}=3h=(\mu'_1rad(\delta)-1).\mu'_1rad(\delta)\Rightarrow \tag{26}$$

$$q-1=2\mu'_1rad(\delta)-2 \tag{27}$$

$$q+1=2\mu'_1rad(\delta) \tag{28}$$

It follows that $(q=x,1=y)$ is a solution of the Diophantine equation:

$$x^2-y^2=N \tag{29}$$

with $N=4\mu'_1rad(\delta)(\mu'_1rad(\delta)-1)=12h>0$. Let $Q(N)$ be the number of the solutions of (29) and $\tau(N)$ is the number of suitable factorization of N, then we announce the following result concerning the solutions of the Diophantine equation (29) (see theorem 27.3 in [7]):

- If $N\equiv 2(\bmod 4)$, then $Q(N)=0$.
- If $N\equiv 1$ or $N\equiv 3(\bmod 4)$, then $Q(N)=[\tau(N)/2]$.
- If $N\equiv 0(\bmod 4)$, then $Q(N)=[\tau(N/4)/2]$.

$[x]$ is the integral part of x for which $[x] \leq x < [x]+1$.

As $N = 4\mu'_1 rad(\delta)(\mu'_1 rad(\delta)-1) \Longrightarrow N \equiv 0(\bmod 4) \Longrightarrow Q(N) = [\tau(N/4)/2]$. As $(q,1)$ is a couple of solutions of the Diophantine equation (29), then $\exists\ d, d'$ positive integers with $d > d'$ and $N = d.d'$ so that :

$$d + d' = 2q \tag{30}$$

$$d - d' = 2.1 = 2 \tag{31}$$

** *1-2-2-1* As $N > 1$, we take $d = N$ and $d' = 1$. It follows:

$$\begin{cases} N+1 = 2q \\ N-1 = 2 \end{cases} \Longrightarrow N = 3 \Longrightarrow \text{then the contradiction with } N \equiv 0(\bmod 4).$$

** *1-2-2-2* Now, we consider the case $d = 2\mu'_1 rad(\delta)(\mu'_1 rad(\delta)-1)$ and $d' = 2$. It follows:

$$\begin{cases} 2\mu'_1 rad(\delta)(\mu'_1 rad(\delta)-1)+2 = 2q \\ 2\mu'_1 rad(\delta)(\mu'_1 rad(\delta)-1)-2 = 2 \end{cases} \Rightarrow 2\mu'_1 rad(\delta)(\mu'_1 rad(\delta)-1) = q+1$$

As $q+1 = 2\mu'_1 rad(\delta)$, we obtain $\mu'_1 rad(\delta) = 2$, then the contradiction with $3|\delta$.

** *1-2-2-3* Now, we consider the case $d = \mu'_1 rad(\delta)(\mu'_1 rad(\delta)-1)$ and $d' = 4$. It follows:

$$\begin{cases} \mu'_1 rad(\delta)(\mu'_1 rad(\delta)-1)+4 = 2q \\ \mu'_1 rad(\delta)(\mu'_1 rad(\delta)-1)-4 = 2 \Rightarrow \mu'_1 rad(\delta)(\mu'_1 rad(\delta)-1) = 6 \end{cases}$$

As $\mu'_1 rad(\delta) > 0 \Longrightarrow \mu'_1 rad(\delta) = 3 \Longrightarrow \mu'_1 = 1, \quad rad(\delta) = 3$ and $q = 5$. From $q^2 = 1+12h$, we obtain $h = 2$. Using the relation (19) $r\mu'_1 - 1 = 3^i h$ as $\mu'_1 = 1, i = 2, h = 2$, it gives $r - 1 = 9h = 18$. As δ is the positive root of the equation (18):

$$Z^2 - 3Z + 3(1-r) = 0 \Longrightarrow \delta = 9 = 3^2$$

But $\delta = 1 + X = 1 + rad(a) \Longrightarrow rad(a) = 8 = 2^3$, then the contradiction.

** *1-2-2-4* Now, let c_{j_0} be a prime integer so that $c_{j_0} | rad\delta$, we consider the case $d = \mu'_1 \dfrac{rad(\delta)}{c_{j_0}}(\mu'_1 rad(\delta)-1)$ and $d' = 4c_{j_0}$. It follows:

$$\begin{cases} \mu'_1 \dfrac{rad(\delta)}{c_{j_0}}(\mu'_1 rad(\delta)-1)+4c_{j_0} = 2q \\ \mu'_1 \dfrac{rad(\delta)}{c_{j_0}}(\mu'_1 rad(\delta)-1)-4c_{j_0} = 2 \end{cases} \Longrightarrow \mu'_1 \frac{rad(\delta)}{c_{j_0}}(\mu'_1 rad(\delta)-1) = 2(1+2c_{j_0}) \Longrightarrow$$

Then the contradiction as the left member is greater than the right member $2(1+2c_{j_0})$.

** *1-2-2-5* Now, we consider the case $d = 4\mu'_1 rad(\delta)$ and $d' = (\mu'_1 rad(\delta)-1)$. It follows:

$$\begin{cases} 4\mu'_1 rad(\delta) + (\mu'_1 rad(\delta)-1) = 2q \\ 4\mu'_1 rad(\delta) - (\mu'_1 rad(\delta)-1) = 2 \end{cases} \Longrightarrow 3\mu'_1 rad(\delta) = 1 \Longrightarrow \text{Then the contradiction.}$$

** *1-2-2-6* Now, we consider the case $d = 2\mu'_1 rad(\delta)$ and $d' = 2(\mu'_1 rad(\delta) - 1)$. It follows:

$$\begin{cases} 2\mu'_1 rad(\delta) + 2(\mu'_1 rad(\delta) - 1) = 2q \Longrightarrow 2\mu'_1 rad(\delta) - 1 = q \\ 2\mu'_1 rad(\delta) - 2(\mu'_1 rad(\delta) - 1) = 2 \Longrightarrow 2 = 2 \end{cases}$$

It follows that this case presents no contradictions a priori.

** *1-2-2-7* $\mu'_1 rad(\delta)$ and $\mu'_1 rad(\delta) - 1$ are coprime, let $\mu'_1 rad(\delta) - 1 = \prod_{j=1}^{j=J} \lambda_j^{\gamma_j}$, we consider the case $d = 2\lambda_{j'}\mu'_1 rad(\delta)$ and $d' = 2\dfrac{\mu'_1 rad(\delta) - 1}{\lambda_{j'}}$. It follows:

$$\begin{cases} 2\lambda_{j'}\mu'_1 rad(\delta) + 2\dfrac{\mu'_1 rad(\delta) - 1}{\lambda_{j'}} = 2q \\ \\ 2\lambda_{j'}\mu'_1 rad(\delta) - 2\dfrac{\mu'_1 rad(\delta) - 1}{\lambda_{j'}} = 2 \end{cases}$$

** *1-2-2-7-1* We suppose that $\gamma_{j'} = 1$. We consider the case $d = 2\lambda_{j'}\mu'_1 rad(\delta)$ and $d' = 2\dfrac{\mu'_1 rad(\delta) - 1}{\lambda_{j'}}$. It follows:

$$\begin{cases} 2\lambda_{j'}\mu'_1 rad(\delta) + 2\dfrac{\mu'_1 rad(\delta) - 1}{\lambda_{j'}} = 2q \\ \\ 2\lambda_{j'}\mu'_1 rad(\delta) - 2\dfrac{\mu'_1 rad(\delta) - 1}{\lambda_{j'}} = 2 \end{cases} \Longrightarrow 4\lambda_{j'}\mu'_1 rad(\delta) = 2(q+1) \Longrightarrow 2\lambda_{j'}\mu'_1 rad(\delta) = q+1$$

But from the equation (28), $q + 1 = 2\mu'_1 rad(\delta)$, then $\lambda_{j'} = 1$, it follows the contradiction.

** *1-2-2-7-2* We suppose that $\gamma_{j'} \geq 2$. We consider the case $d = 2\lambda_{j'}^{\gamma_{j'} - r'_{j'}}\mu'_1 rad(\delta)$ and $d' = 2\dfrac{\mu'_1 rad(\delta) - 1}{\lambda_{j'}^{r'_{j'}}}$. It follows:

$$\begin{cases} 2\lambda_{j'}^{\gamma_{j'} - r'_{j'}}\mu'_1 rad(\delta) + 2\dfrac{\mu'_1 rad(\delta) - 1}{\lambda_{j'}^{r'_{j'}}} = 2q \\ \\ 2\lambda_{j'}^{\gamma_{j'} - r'_{j'}}\mu'_1 rad(\delta) - 2\dfrac{\mu'_1 rad(\delta) - 1}{\lambda_{j'}^{r'_{j'}}} = 2 \end{cases} \Longrightarrow 4\lambda_{j'}^{\gamma_{j'} - r'_{j'}}\mu'_1 rad(\delta) = 2(q+1)$$

$$\Longrightarrow 2\lambda_{j'}^{\gamma_{j'} - r'_{j'}}\mu'_1 rad(\delta) = q + 1$$

As above, it follows the contradiction. It is trivial that the other cases for more factors $\prod_{j"} \lambda_{j"}^{\gamma_{j"} - r"_{j"}}$ give also contradictions.

** *1-2-2-8* Now, we consider the case $d = 4(\mu_1' rad(\delta) - 1)$ and $d' = \mu_1' rad(\delta)$, we have $d > d'$. It follows:

$$\left\{\begin{array}{l} 4(\mu_1' rad(\delta) - 1) + \mu_1' rad(\delta) = 2q \Rightarrow 5\mu_1' rad(\delta) = 2(q+2) \\ 4(\mu_1' rad(\delta) - 1) - \mu_1' rad(\delta) = 2 \Rightarrow \mu_1' rad(\delta) = 2 \end{array}\right. \Rightarrow \left\{\begin{array}{l} \text{Then the contradiction as} \\ 3|\delta. \end{array}\right.$$

** *1-2-2-9* Now, we consider the case $d = 4u(\mu_1' rad(\delta) - 1)$ and $d' = \dfrac{\mu_1' rad(\delta)}{u}$, where $u > 1$ is an integer divisor of $\mu_1' rad(\delta)$. We have $d > d'$ and:

$$\left\{\begin{array}{l} 4u(\mu_1' rad(\delta) - 1) + \dfrac{\mu_1' rad(\delta)}{u} = 2q \\ \\ 4u(\mu_1' rad(\delta) - 1) - \dfrac{\mu_1' rad(\delta)}{u} = 2 \end{array}\right. \Longrightarrow 2u(\mu_1' rad(\delta) - 1) = \mu_1' rad(\delta)$$

Then the contradiction as $\mu_1' rad(\delta)$ and $(\mu_1' rad(\delta) - 1)$ are coprime.

In conclusion, we have found only one case (** *1-2-2-6* above) where there is no contradictions *a priori*. As $\tau(N)$ is large and also $[\tau(N/4)/2]$, it follows the contradiction with $Q(N) \leq 1$ and the hypothesis $(\mu_1, \mu_2) \neq 1$ is false.

** *2-* We suppose that $(\mu_1, \mu_2) = 1$.

From the equation $r\mu_1 = \delta^2 - 3X$ and the condition $rad(a) = X > rad^{1.63/1.37}(c) \Longleftrightarrow \delta - 1 = X > rad^{1.19}(c)$, we obtain the following inequality:

$$\begin{gathered} \delta - 1 > (r.rad(\delta))^{1.19} \Longrightarrow -3(\delta - 1) < -3r.rad(\delta).(r.rad(\delta))^{0.19} \Longrightarrow \\ r\mu_1 = \delta^2 - 3(\delta - 1) < (r.rad(\delta))^2 - 3r.rad(\delta).(r.rad(\delta))^{0.19} \Longrightarrow \\ \mu_1 < r.rad^2(\delta) - 3.rad(\delta).(r.rad(\delta))^{0.19} \Longrightarrow \end{gathered}$$

$$\mu_1 < r.rad^2(\delta)\left(1 - \frac{3}{(r.rad(\delta))^{0.81}}\right) \tag{32}$$

As $a = rad^3(a) < c$, we can write:

$$rad^3(a) < \mu_1\mu_2 rad(c) < \mu_2.rad(\delta).rad^2(c)\left(1 - \frac{3}{(r.rad(\delta))^{0.81}}\right)$$

but $(r, rad(\delta)) = 1$, $r.rad(\delta) \geq 6 \Longrightarrow (r.rad(\delta))^{0.81} \geq (6^{0.81} \approx 4.26)$ and $\delta = \mu_2.rad(\delta)$, it follows:

$$rad^3(a) < \mu_1\mu_2 rad(c) < \mu_2.rad(\delta).rad^2(c) \Longrightarrow rad^3(a) < \delta.rad^2(c) < 1.6rad(a).rad^2(c)$$

As $rad(a) > (rad^{1.62/1.37}(c) = rad^{1.19}(c)) \Longrightarrow rad^{1.19}(c) < rad(a) < 1.27rad(c)$, then we obtain:

$$rad^{1.19}(c) < 1.27rad(c) \Longrightarrow rad(c) < 3.5 \Longrightarrow rad(c) \leq 3, \text{ but } rad(c) = r.rad(\delta) \geq 6$$

Then the contradiction.

It follows that the case $\mu_c > rad^{2.26}(c) \Rightarrow c > rad^{3.26}(c)$ and $a = rad^3(a)$ is impossible.

I-3-2-2- We consider the case $\mu_a = rad^2(a) \Longrightarrow a = rad^3(a)$ and $c = a + b$. Then, we obtain that $X = rad(a)$ is a solution in positive integers of the equation:

$$X^3 + 1 = \bar{c} \tag{33}$$

with $\bar{c} = c - b + 1 = a + 1 \Longrightarrow (\bar{c}, a) = 1$. We obtain the same result as of the case **I-3-2-1-** studied above considering $rad(a) > rad^{\frac{1.63}{1.37}}(\bar{c})$.

I-3-2-3- We suppose $\mu_c > rad^{2.26}(c) \Rightarrow c > rad^{3.26}(c)$ and c large and $\mu_a < rad^2(a)$, we consider $c = a+b, b \geq 1$. Then $c = rad^3(c)+h, h > rad^3(c)$, h a positive integer and we can write $a+l = rad^3(a)$, $l > 0$. Then we obtain :

$$rad^3(c) + h = rad^3(a) - l + b \Longrightarrow rad^3(a) - rad^3(c) = h + l - b > 0 \tag{34}$$

as $rad(a) > rad^{\frac{1.63}{1.37}}(c)$. We obtain the equation:

$$rad^3(a) - rad^3(c) = h + l - b = m > 0 \tag{35}$$

Let $X = rad(a) - rad(c)$, then X is an integer root of the polynomial $H(X)$ defined as:

$$H(X) = X^3 + 3rad(ac)X - m = 0 \tag{36}$$

To resolve the above equation, we denote $X = u + v$, It follows that u^3, v^3 are the roots of the polynomial $G(t)$ given by:

$$G(t) = t^2 - mt - rad^3(ac) = 0 \tag{37}$$

The discriminant of $G(t)$ is $\Delta = m^2 + 4rad^3(ac) = \alpha^2, \quad \alpha > 0$. As $m = rad^3(a) - rad^3(c) > 0$, we obtain that $\alpha = rad^3(a) + rad^3(c) > 0$, then from the expression of the discriminant Δ, it follows that the couple $(\alpha = x, m = y)$ is a solution of the Diophantine equation:

$$x^2 - y^2 = N \tag{38}$$

with $N = 4rad^3(ac) = 4rad^3(a).rad^3(c) > 0$. Here, we will use the same method that is given in the above sub-paragraph ** *1-2-2-* of the paragraph **I-3-2-1-2-5-**. We have the two terms $rad^3(a)$ and $rad^3(c)$ coprime. As (α, m) is a couple of solutions of the Diophantine equation (38) and $\alpha > m$, then $\exists\ d, d'$ positive integers with $d > d'$ and $N = d.d'$ so that :

$$d + d' = 2\alpha \tag{39}$$

$$d - d' = 2m \tag{40}$$

I-3-2-3-1- Let us consider the case $d = 2rad^3(a)$, $d' = 2rad^3(c)$. It follows:

$$\begin{cases} 2rad^3(a) + 2rad^3(c) = 2\alpha \Longrightarrow \alpha = rad^3(a) + rad^3(c) \\ 2rad^3(a) - 2rad^3(c) = 2m \Longrightarrow m = rad^3(a) - rad^3(c) \end{cases}$$

It follows that this case presents *a priori* no contradictions.

I-3-2-3-2- Now, we consider for example, the case $d = 4rad^3(a)$ and $d' = rad^3(c) \Longrightarrow d > d'$. We rewrite the equations (39-40):

$$4rad^3(a) + rad^3(c) = 2(rad^3(a) + rad^3(c)) \Rightarrow 2rad^3(a) = rad^3(c))$$
$$4rad^3(a) - rad^3(c) = 2(rad^3(a) - rad^3(c)) \Longrightarrow 2rad^3(a) = -rad^3(c))$$

Then the contradiction.

I-3-2-3-3- We consider the case $d = 4rad^3(c)rad^3(a)$ and $d' = 1 \Longrightarrow d > d'$. We rewrite the equations (39-40):

$$4rad^3(c)rad^3(a) + 1 = 2(rad^3(c) + rad^3(a)) \Longrightarrow$$
$$2(2rad^3(c)rad^3(a) - rad^3(c) - rad^3(a)) = -1 \Rightarrow \text{ a contradiction}$$
$$4rad^3(c)rad^3(a) - 1 = 2(rad^3(c) - rad^3(a))$$

Then the contradiction.

I-3-2-3-4- Let c_1 be the first factor of $rad(c)$. We consider the case $d = 4c_1rad^3(a)$ and $d' = \dfrac{rad^3(c)}{c_1} \Longrightarrow d > d'$. We rewrite the equation (39):

$$4c_1rad^3(a) + \frac{rad^3(c)}{c_1} = 2(rad^3(a) + rad^3(c)) \Rightarrow$$
$$2rad^3(a)(2c_1 - 1) = \frac{rad^3(c)}{c_1}(2c_1 - 1) \Rightarrow 2rad^3(a) = rad^2(c).\frac{rad(c)}{c_1}$$

$c_1 = 2$ or not, there is a contradiction with a, c coprime.

The other cases of the expressions of d and d' not coprime so that $N = d.d'$ give also contradictions.

Let $Q(N)$ be the number of the solutions of (38), as $N \equiv 0 (\bmod 4)$, then $Q(N) = [\tau(N/4)/2]$. From the study of the cases above, we obtain that $Q(N) \leq 1$ is $\ll [(\tau(N)/4)/2]$. It follows the contradiction.

Then the cases $\mu_a \leq rad^2(a)$ and $c > rad^{3.26}(c)$ are impossible.

II- We suppose that $rad^{1.63}(c) < \mu_c \leq rad^2(c)$ and $\mu_a > rad^{1.63}(a)$:

II-1- Case $rad(c) < rad(a)$: As $c \leq rad^3(c) = rad^{1.63}(c).rad^{1.37}(c) \Longrightarrow c < rad^{1.63}(c).rad^{1.37}(a) < rad^{1.63}(ac) < rad^{1.63}(abc) \Longrightarrow \boxed{c < R^{1.63}}$.

II-2- Case $rad(a) < rad(c) < rad^{\frac{1.63}{1.37}}(a)$:
As $c \leq rad^3(c) \leq rad^{1.63}(c).rad^{1.37}(c) \Longrightarrow c < rad^{1.63}(c).rad^{1.63}(a) < rad^{1.63}(abc) \Longrightarrow \boxed{c < R^{1.63}}$.

II-3- Case $rad^{\frac{1.63}{1.37}}(a) < rad(c)$:

II-3-1- We suppose $rad^{1.63}(a) < \mu_a \leq rad^{2.26}(a) \Longrightarrow a \leq rad^{1.63}(a).rad^{1.63}(a) \Longrightarrow a < rad^{1.63}(a).rad^{1.37}(c) \Longrightarrow c = a + b < 2a < 2rad^{1.63}(a).rad^{1.63}(c) < rad^{1.63}(abc) \Longrightarrow c < R^{1.63} \Longrightarrow \boxed{c < R^{1.63}}$.

II-3-2- We suppose $\mu_a > rad^{2.26}(a) \Longrightarrow a > rad^{3.26}(a)$ and $\mu_c \leq rad^2(c)$. Using the same method as it was explicated in the paragraphs **I-3-2-** (permuting a, c see in Appendix **II'-3-2-**), we arrive at a contradiction. It follows that the cases $\mu_c \leq rad^2(c)$ and $\mu_a > rad^{2.26}(a)$ are impossible.

2.2.3. *Case $\mu_a > rad^{1.63}(a)$ and $\mu_c > rad^{1.63}(c)$:* Taking into account the cases studied above, it remains to see the following two cases:
- $\mu_c > rad^2(c)$ and $\mu_a > rad^{1.63}(a)$,
- $\mu_a > rad^2(a)$ and $\mu_c > rad^{1.63}(c)$.

III- We suppose $\mu_c > rad^2(c)$ and $\mu_a > rad^{1.63}(a) \Longrightarrow c > rad^3(c)$ and $a > rad^{2.63}(a)$. We can write $c = rad^3(c) + h$ and $a = rad^3(a) + l$ with h a positive integer and $l \in \mathbb{Z}$.

III-1- We suppose $rad(c) < rad(a)$. We obtain the equation:

$$rad^3(a) - rad^3(c) = h - l - b = m > 0 \tag{41}$$

Let $X = rad(a) - rad(c)$, from the above equation, X is a real root of the polynomial:

$$H(X) = X^3 + 3rad(ac)X - m = 0 \tag{42}$$

As above, to resolve (42), we denote $X = u + v$, It follows that u^3, v^3 are the roots of the polynomial $G(t)$ given by :

$$G(t) = t^2 - mt - rad^3(ac) = 0 \tag{43}$$

The discriminant of $G(t)$ is:

$$\Delta = m^2 + 4rad^3(ac) = \alpha^2, \quad \alpha > 0 \tag{44}$$

As $m = rad^3(a) - rad^3(c) > 0$, we obtain that $\alpha = rad^3(a) + rad^3(c) > 0$, then from the equation (44), it follows that $(\alpha = x, m = y)$ is a solution of the Diophantine equation:

$$x^2 - y^2 = N \tag{45}$$

with $N = 4rad^3(ac) > 0$. Let $Q(N)$ be the number of the solutions of (45) and $\tau(N)$ is the number of suitable factorization of N, and using the same method as in the paragraph **I-3-2-3-** above, we obtain a contradiction.

III-2- We suppose $rad(a) < rad(c)$. We obtain the equation:

$$rad^3(c) - rad^3(a) = b + l - h = m > 0 \tag{46}$$

Let X be the variable $X = rad(c) - rad(a)$, we use the similar calculations as in the paragraph above **I-3-2-3-** permuting c, a, we find a contradiction.

It follows that the case $\mu_c > rad^2(c)$ and $\mu_a > rad^{1.63}(a)$ is impossible.

IV - We suppose $\mu_a > rad^2(a)$ and $\mu_c > rad^{1.63}(c)$, we obtain $a > rad^3(a)$ and $c > rad^{2.63}(c)$. We can write $a = rad^3(a) + h$ and $c = rad^3(c) + l$ with h a positive integer and $l \in \mathbb{Z}$.

The calculations are similar to those in the cases of the paragraph **III**. We obtain a contradiction.

It follows that the case $\mu_c > rad^{1.63}(c)$ and $\mu_a > rad^2(a)$ is impossible.

All possible cases are discussed. $\square$

We can state the following important theorem:

Theorem 2.2. *Let a, b, c positive integers relatively prime with $c = a + b$, then $c < rad^{1.63}(abc)$.*

From the theorem above, we can announce also:

Corollary 2.2.1. *Let a, b, c positive integers relatively prime with $c = a+b$, then the conjecture $c < rad^2(abc)$ is true.*

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Appendix

II'-3-2- We suppose $\mu_a > rad^{2.26}(a) \Longrightarrow a > rad^{3.26}(a)$.

II'-3-2-1- We consider the case $\mu_c = rad^2(c) \Longrightarrow c = rad^3(c)$ and $c = a+1$. Then, we obtain that $Y = rad(c)$ is a solution in positive integers of the equation:

$$Y^3 - 1 = a \tag{47}$$

II'-3-2-1-1- We suppose that $a = rad^n(a)$ with $n \geq 4$, we obtain the equation:

$$rad^3(c) - rad^n(a) = 1 \tag{48}$$

But the solutions of the Catalan equation [5] $x^p - y^q = 1$ where the unknowns x, y, p and q take integer values, all ≥ 2, has only one solution $(x, y, p, q) = (3, 2, 2, 3)$, but the solution of the equation (48) are $(rad(c) = 3, rad(a) = 2, 3 \neq 2, n \geq 4)$, it follows the contradiction with $n \geq 4$ and the case $a = rad^n(a), n \geq 4$ is to reject.

II'-3-2-1-2- In the following, we will study the cases $\mu_a = A.rad^n(a)$ with $rad(a) \nmid A, n \geq 0$. The above equation (47) can be written as :

$$(Y-1)(Y^2+Y+1) = a \tag{49}$$

Let δ one divisor of a so that :

$$Y - 1 = \delta \tag{50}$$

$$Y^2 + Y + 1 = \frac{a}{\delta} = m = \delta^2 + 3Y \tag{51}$$

We recall that $rad(c) > rad^{\frac{1.63}{1.37}}(a)$.

II'-3-2-1-2-1- We suppose $\delta = l.rad(a)$. We have $\delta = l.rad(a) < a = \mu_a.rad(a) \Longrightarrow l < \mu_a$. As δ is a divisor of a, then l is a divisor of μ_a, $\dfrac{a}{\delta} = \dfrac{\mu_a rad(a)}{l.rad(a)} = \dfrac{\mu_a}{l} = m = \delta^2 + 3Y$, then $\mu_a = l.m$. From $\mu_a = l(\delta^2+3Y)$, we obtain:

$$m = l^2 rad^2(a) + 3rad(c) \Longrightarrow 3rad(c) = m - l^2 rad^2(a)$$

A'- Case $3|m \Longrightarrow m = 3m'$, $m' > 1$: As $\mu_a = ml = 3m'l \Longrightarrow 3|rad(a)$ and $(rad(a), m')$ not coprime. We obtain:

$$rad(c) = m' - l^2 rad(a).\frac{rad(a)}{3}$$

It follows that a, c are not coprime, then the contradiction.

B' - Case $m = 3 \Longrightarrow \mu_a = 3l \Longrightarrow a = 3lrad(a) = 3\delta = \delta(\delta^2+3Y) \Longrightarrow \delta^2 = 3(1-Y) = -3\delta < 0$, then the contradiction.

II'-3-2-1-2-2- We suppose $\delta = l.rad^2(a), l \geq 2$. If $n = 0$ then $\mu_a = A$ and from the equation above (51):

$$m = \frac{a}{\delta} = \frac{\mu_a.rad(a)}{lrad^2(a)} = \frac{A.rad(a)}{lrad^2(a)} = \frac{A}{lrad(a)} \Rightarrow rad(a)|A$$

It follows the contradiction with the hypothesis above $rad(a) \nmid A$.

II'-3-2-1-2-3- We suppose $\delta = lrad^2(a), l \geq 2$ and in the following $n > 0$. As $m = \dfrac{a}{\delta} = \dfrac{\mu_a.rad(a)}{lrad^2(a)} = \dfrac{\mu_a}{lrad(a)}$, if $lrad(a) \nmid \mu_a$ then the case is to reject. We suppose $lrad(a)|\mu_a \Longrightarrow \mu_a = m.lrad(a)$, with $m, rad(a)$ not coprime, then $\dfrac{a}{\delta} = m = \delta^2 + 3rad(c)$.

C' - Case $m = 1 = a/\delta \Longrightarrow \delta^2 + 3rad(c) = 1$, then the contradiction.

D' - Case $m = 3$, we obtain $3(1 - rad(c)) = \delta^2 \Longrightarrow \delta^2 < 0$. Then the contradiction.

E' - Case $m \neq 1, 3$, we obtain: $3rad(c) = m - l^2 rad^4(a) \Longrightarrow rad(a)$ and $rad(c)$ are not coprime. Then the contradiction.

II'-3-2-1-2-4- We suppose $\delta = l.rad^n(a), l \geq 2$ with $n \geq 3$. From $a = \mu_a.rad(a) = lrad^n(a)(\delta^2 + 3rad(c))$, we denote $m = \delta^2 + 3rad(c) = \delta^2 + 3Y$.

F' - As seen above (paragraphs C',D'), the cases $m = 1$ and $m = 3$ give contradictions, it follows the reject of these cases.

G' - Case $m \neq 1, 3$. Let q be a prime that divides m (q can be equal to m), it follows $q|\mu_a \Longrightarrow q = a_{j'_0} \Longrightarrow a_{j'_0}|\delta^2 \Longrightarrow a_{j'_0}|3rad(c)$. Then $rad(a)$ and

$rad(c)$ are not coprime. It follows the contradiction.

II'-3-2-1-2-5- We suppose $\delta = \prod_{j \in J_1} a_j^{\beta_j}$, $\beta_j \geq 1$ with at least one $j_0 \in J_1$ with:

$$\beta_{j_0} \geq 2, \quad rad(a) \nmid \delta \tag{52}$$

We can write:

(53)

$$\delta = \mu_\delta.rad(\delta), \quad rad(a) = r.rad(\delta), \quad r > 1, \quad (r, rad(\delta)) = 1 \Rightarrow (r, \mu_\delta) = 1$$

Then, we obtain:

$$a = \mu_a.rad(a) = \mu_a.r.rad(\delta) = \delta(\delta^2 + 3Y) = \mu_\delta.rad(\delta)(\delta^2 + 3Y) \Longrightarrow$$

$$r.\mu_a = \mu_\delta(\delta^2 + 3Y) \tag{54}$$

- We suppose $\mu_a = \mu_\delta \Longrightarrow r = \delta^2 + 3Y = (\mu_a.rad(\delta))^2 + 3Y$. As $\delta < \delta^2 + 3Y \Longrightarrow r > \delta \Longrightarrow rad(a) > r > (\mu_a.rad(\delta) = A.rad^n(a)rad(\delta)) \Longrightarrow 1 > A.rad^{n-1}(\delta)$, then the contradiction.

- We suppose $\mu_a < \mu_\delta$. As $rad(c) = \mu_\delta rad(\delta) + 1$, we obtain:

$$rad(c) > \mu_a.rad(\delta) + 1 > 0 \Longrightarrow rad(ac) > a.rad(\delta) + rad(a) > 0$$

As $c = 1 + a$ and we consider the cases $c > rad(ac)$, then:

$$c > rad(ac) > a.rad(\delta) + rad(a) > 0 \Longrightarrow a + 1 \geq a.rad(\delta) + rad(a) > 0 \Longrightarrow$$

$$a \geq a.rad(\delta) + rad(\delta) \Longrightarrow 1 \geq rad(\delta) + \frac{rad(a)}{a} > 0, \quad rad(\delta) \geq 2 \Longrightarrow \text{The contradiction}$$

- We suppose $\mu_a > \mu_\delta$. In this case, from the equation (14) and as $(r, \mu_\delta) = 1$, it follows we can write:

$$\mu_a = \mu_1.\mu_2, \quad \mu_1, \mu_2 > 1 \tag{55}$$

$$a = \mu_a rad(a) = \mu_1.\mu_2.r.rad(\delta) = \delta.(\delta^2 + 3Y) \tag{56}$$

$$\text{so that} \quad r.\mu_1 = \delta^2 + 3Y, \quad \mu_2 = \mu_\delta \Longrightarrow \delta = \mu_2.rad(\delta) \tag{57}$$

** *1-* We suppose $(\mu_1, \mu_2) \neq 1$, then $\exists\, a_{j_0}$ so that $a_{j_0}|\mu_1$ and $a_{j_0}|\mu_2$. But $\mu_\delta = \mu_2 \Rightarrow a_{j_0}^2|\delta$. From $3Y = r\mu_1 - \delta^2 \Longrightarrow a_{j_0}|3Y \Longrightarrow a_{j_0}|Y$ or $a_{j_0} = 3$.

- If $a_{j_0}|(Y = rad(c))$, it follows the contradiction with $(c, a) = 1$.
- If $a_{j_0} = 3$. We have $r\mu_1 = \delta^2 + 3Y = \delta^2 + 3(\delta + 1) \Longrightarrow \delta^2 + 3\delta + 3 - r.\mu_1 = 0$.

As $3|\mu_1 \Longrightarrow \mu_1 = 3^k\mu'_1, 3 \nmid \mu'_1, k \geq 1$, we obtain:

$$\delta^2 + 3\delta + 3(1 - 3^{k-1}r\mu'_1) = 0 \tag{58}$$

** *1-1-* We consider the case $k > 1 \Longrightarrow 3 \nmid (1 - 3^{k-1}r\mu'_1)$. Let us recall the Eisenstein criterion [6]:

Theorem 2.3. ***(Eisenstein Criterion)*** *Let $f = a_0 + \cdots + a_nX^n$ be a polynomial $\in \mathbb{Z}[X]$. We suppose that $\exists p$ a prime number so that $p \nmid a_n$, $p|a_i$, $(0 \leq i \leq n-1)$, and $p^2 \nmid a_0$, then f is irreducible in $\mathbb{Q}$.*

We apply Eisenstein criterion to the polynomial $R(Z)$ given by:

$$R(Z) = Z^2 + 3Z + 3(1 - 3^{k-1}r\mu'_1) \tag{59}$$

then:

- $3 \nmid 1$, - $3|(+3)$,- $3|3(1 - 3^{k-1}r\mu'_1)$, and - $3^2 \nmid 3(1 - 3^{k-1}r\mu'_1)$.

It follows that the polynomial $R(Z)$ is irreducible in $\mathbb{Q}$, then, the contradiction with $R(\delta)=0$.

** *1-2-* We consider the case $k=1$, then $\mu_1=3\mu_1'$ and $(\mu_1',3)=1$, we obtain:

$$\delta^2+3\delta+3(1-r\mu_1')=0 \tag{60}$$

** *1-2-1-* We consider that $3\nmid(1-r.\mu_1')$, we apply the same Eisenstein criterion to the polynomial $R'(Z)$ given by:

$$R'(Z)=Z^2+3Z+3(1-r\mu_1')$$

and we find a contradiction with $R'(\delta)=0$.

** *1-2-2-* We consider that:

$$3|(1-r.\mu_1') \Longrightarrow r\mu_1'-1=3^i.h,\ i\geq 1,\ 3\nmid h, h\in\mathbb{N}^* \tag{61}$$

δ is an integer root of the polynomial $R'(Z)$:

$$R'(Z)=Z^2+3Z+3(1-r\mu_1')=0 \tag{62}$$

The discriminant of $R'(Z)$ is:

$$\Delta=3^2+3^{i+1}\times 4.h$$

As the root δ is an integer, it follows that $\Delta=t^2>0$ with t a positive integer. We obtain:

$$\Delta=3^2(1+3^{i-1}\times 4h)=t^2 \tag{63}$$

$$\Longrightarrow 1+3^{i-1}\times 4h=q^2>1, q\in\mathbb{N}^* \tag{64}$$

As $\mu_\delta=\mu_2$ and $3|\mu_2\Longrightarrow \mu_2=3\mu_2'$, then we can write the equation (60) as :

$$\delta(\delta+3)=3^{i+1}.h\Longrightarrow 3^3\mu_2'\frac{rad(\delta)}{3}.\big(\mu_2'rad(\delta)+1\big)=3^{i+1}.h\Longrightarrow \tag{65}$$

$$\mu_2'\frac{rad(\delta)}{3}.\big(\mu_2'rad(\delta)+1\big)=h \tag{66}$$

We obtain $i=2$ and $q^2=1+12h=1+4\mu_2'rad(\delta)(\mu_2'rad(\delta)+1)$. Then, q satisfies :

$$q^2-1=12h=4\mu_2'rad(\delta)(\mu_2'rad(\delta)+1)\Longrightarrow \tag{67}$$

$$\tfrac{(q-1)}{2}.\tfrac{(q+1)}{2}=3h=\mu_2'rad(\delta)(\mu_2'rad(\delta)+1).\Rightarrow \tag{68}$$

$$q+1=2\mu_2'rad(\delta)+2 \tag{69}$$

$$q-1=2\mu_2'rad(\delta) \tag{70}$$

It follows that $(q=x,1=y)$ is a solution of the Diophantine equation:

$$x^2-y^2=N \tag{71}$$

with $N=4\mu_2'rad(\delta)(\mu_2'rad(\delta)+1)=12h>0$. Let $Q(N)$ be the number of the solutions of (71) and $\tau(N)$ is the number of suitable factorization of N, then we announce the following result concerning the solutions of the Diophantine equation (71) (see theorem 27.3 in [7]):

- If $N\equiv 2 (\bmod 4)$, then $Q(N)=0$.
- If $N\equiv 1$ or $N\equiv 3 (\bmod 4)$, then $Q(N)=[\tau(N)/2]$.
- If $N\equiv 0 (\bmod 4)$, then $Q(N)=[\tau(N/4)/2]$.

$[x]$ is the integral part of x for which $[x] \leq x < [x]+1$.

As $N = 4\mu'_2 rad(\delta)(\mu'_2 rad(\delta)+1) \Longrightarrow N \equiv 0 (\bmod 4) \Longrightarrow Q(N) = [\tau(N/4)/2]$.
As $(q,1)$ is a couple of solutions of the Diophantine equation (71), then $\exists\ d, d'$ positive integers with $d > d'$ and $N = d.d'$ so that :

$$d + d' = 2q \tag{72}$$

$$d - d' = 2.1 = 2 \tag{73}$$

** *1-2-2-1* As $N > 1$, we take $d = N$ and $d' = 1$. It follows:

$$\left\{ \begin{array}{l} N+1 = 2q \\ N-1 = 2 \end{array} \right. \Longrightarrow N = 3 \Longrightarrow \text{then the contradiction with } N \equiv 0 (\bmod 4).$$

** *1-2-2-2* Now, we consider the case $d = 2\mu'_2 rad(\delta)(\mu'_2 rad(\delta)+1)$ and $d' = 2$. It follows:

$$\left\{ \begin{array}{l} 2\mu'_2 rad(\delta)(\mu'_2 rad(\delta)+1)+2 = 2q \\ 2\mu'_2 rad(\delta)(\mu'_2 rad(\delta)+1)-2 = 2 \end{array} \right. \Rightarrow \mu'_2 rad(\delta)(\mu'_2 rad(\delta)+1) = q-1$$

As $q - 1 = 2\mu'_2 rad(\delta)$, we obtain $\mu'_2 rad(\delta) = 1$, then the contradiction.

** *1-2-2-3* Now, we consider the case $d = \mu'_2 rad(\delta)(\mu'_2 rad(\delta)+1)$ and $d' = 4$. It follows:

$$\left\{ \begin{array}{l} \mu'_2 rad(\delta)(\mu'_2 rad(\delta)+1)+4 = 2q \\ \mu'_2 rad(\delta)(\mu'_2 rad(\delta)+1)-4 = 2 \Rightarrow \mu'_2 rad(\delta)(\mu'_2 rad(\delta)+1) = 6 \end{array} \right.$$

As $\mu'_2 rad(\delta) \geq 2 \Longrightarrow \mu'_2 rad(\delta) = 2 \Longrightarrow \mu'_2 = 1 \Rightarrow \mu_2 = 3 = \mu_\delta$ and $rad(\delta) = 2$ but $3 \nmid 2$, then the contradiction.

** *1-2-2-4* Now, let a_{j_0} be a prime integer so that $a_{j_0} | rad\delta$, we consider the case $d = \mu'_2 \dfrac{rad(\delta)}{a_{j_0}}(\mu'_2 rad(\delta)+1)$ and $d' = 4a_{j_0}$. It follows:

$$\left\{ \begin{array}{l} \mu'_2 \dfrac{rad(\delta)}{a_{j_0}}(\mu'_2 rad(\delta)+1)+4a_{j_0} = 2q \\ \mu'_2 \dfrac{rad(\delta)}{a_{j_0}}(\mu'_2 rad(\delta)+1)-4a_{j_0} = 2 \end{array} \right. \Longrightarrow \mu'_2 \frac{rad(\delta)}{a_{j_0}}(\mu'_2 rad(\delta)+1) = 2(1+2a_{j_0}) \Longrightarrow$$

Then the contradiction as the left member is greater than the right member $2(1+2a_{j_0})$.

** *1-2-2-5* Now, we consider the case $d = 4\mu'_2 rad(\delta)$ and $d' = (\mu'_2 rad(\delta)+1)$. It follows:

$$\left\{ \begin{array}{l} 4\mu'_2 rad(\delta) + (\mu'_2 rad(\delta)+1) = 2q \\ 4\mu'_2 rad(\delta) - (\mu'_2 rad(\delta)+1) = 2 \end{array} \right. \Longrightarrow 3\mu'_2 rad(\delta) = 3 \Longrightarrow \text{Then the contradiction.}$$

** *1-2-2-6* Now, we consider the case $d = 2(\mu'_2 rad(\delta)+1)$ and $d = 2\mu'_2 rad(\delta)$. It follows:

$$\left\{ \begin{array}{l} 2(\mu'_2 rad(\delta)+1) + 2\mu'_2 rad(\delta) = 2q \Longrightarrow 2\mu'_2 rad(\delta)+1 = q \\ 2(\mu'_2 rad(\delta)+1) - 2\mu'_2 rad(\delta) = 2 \Longrightarrow 2 = 2 \end{array} \right.$$

It follows that this case presents no contradictions *a prior.*

** *1-2-2-7* $\mu'_2rad(\delta)$ and $\mu'_2rad(\delta)+1$ are coprime, let $\mu'_2rad(\delta)+1 = \prod_{j=1}^{j=J} \lambda_j^{\gamma_j}$, we consider the case $d = 2\lambda_{j'}\mu'_2rad(\delta)$ and $d' = 2\dfrac{\mu'_2rad(\delta)+1}{\lambda_{j'}}$. It follows:

$$\begin{cases} 2\lambda_{j'}\mu'_2rad(\delta) + 2\dfrac{\mu'_2rad(\delta)+1}{\lambda_{j'}} = 2q \\ \\ 2\lambda_{j'}\mu'_2rad(\delta) - 2\dfrac{\mu'_2rad(\delta)+1}{\lambda_{j'}} = 2 \end{cases}$$

** *1-2-2-7-1* We suppose that $\gamma_{j'} = 1$. We consider the case $d = 2\lambda_{j'}\mu'_2rad(\delta)$ and $d' = 2\dfrac{\mu'_2rad(\delta)+1}{\lambda_{j'}}$. It follows:

$$\begin{cases} 2\lambda_{j'}\mu'_1rad(\delta) + 2\dfrac{\mu'_1rad(\delta)-1}{\lambda_{j'}} = 2q \\ \\ 2\lambda_{j'}\mu'_1rad(\delta) - 2\dfrac{\mu'_1rad(\delta)-1}{\lambda_{j'}} = 2 \end{cases} \Longrightarrow 4\lambda_{j'}\mu'_1rad(\delta) = 2(q+1) \Longrightarrow 2\lambda_{j'}\mu'_1rad(\delta) = q+1$$

But from the equation (28), $q+1 = 2\mu'_1rad(\delta)$, then $\lambda_{j'} = 1$, it follows the contradiction.

** *1-2-2-7-2* We suppose that $\gamma_{j'} \geq 2$. We consider the case $d = 2\lambda_{j'}^{\gamma_{j'}-r'_{j'}}\mu'_2rad(\delta)$ and $d' = 2\dfrac{\mu'_2rad(\delta)+1}{\lambda_{j'}^{r'_{j'}}}$. It follows:

$$\begin{cases} 2\lambda_{j'}^{\gamma_{j'}-r'_{j'}}\mu'_2rad(\delta) + 2\dfrac{\mu'_2rad(\delta)+1}{\lambda_{j'}^{r'_{j'}}} = 2q \\ \\ 2\lambda_{j'}^{\gamma_{j'}-r'_{j'}}\mu'_2rad(\delta) - 2\dfrac{\mu'_2rad(\delta)+1}{\lambda_{j'}^{r'_{j'}}} = 2 \end{cases} \Longrightarrow 4\lambda_{j'}^{\gamma_{j'}-r'_{j'}}\mu'_2rad(\delta) = 2(q+1)$$

$$\Longrightarrow 2\lambda_{j'}^{\gamma_{j'}-r'_{j'}}\mu'_2rad(\delta) = q+1$$

As above, it follows the contradiction. It is trivial that the other cases for more factors $\prod_{j"} \lambda_{j"}^{\gamma_{j"}-r"_{j"}}$ give also contradictions.

** *1-2-2-8* Now, we consider the case $d = 4(\mu'_2rad(\delta)+1)$ and $d' = \mu'_2rad(\delta)$, we have $d > d'$. It follows:

$$\begin{cases} 4(\mu'_2rad(\delta)+1) + \mu'_2rad(\delta) = 2q \Rightarrow 5\mu'_2rad(\delta) = 2(q+2) \\ 4(\mu'_2rad(\delta)+1) - \mu'_2rad(\delta) = 2 \Rightarrow \mu'_2rad(\delta) = 2 \end{cases} \Rightarrow \begin{cases} \text{Then the contradiction as} \\ 3|\delta. \end{cases}$$

** *1-2-2-9* Now, we consider the case $d = 4u(\mu'_2rad(\delta)+1)$ and $d' = \dfrac{\mu'_2rad(\delta)}{u}$, where $u > 1$ is an integer divisor of $\mu'_2rad(\delta)$. We have $d > d'$

and:

$$\begin{cases} 4u(\mu'_2 rad(\delta)+1) + \dfrac{\mu'_2 rad(\delta)}{u} = 2q \\ \\ 4u(\mu'_2 rad(\delta)+1) - \dfrac{\mu'_2 rad(\delta)}{u} = 2 \end{cases} \Longrightarrow 2u(\mu'_2 rad(\delta)+1) = \mu'_2 rad(\delta)+1 \Rightarrow 2u = 1$$

Then the contradiction.

In conclusion, we have found only one case (** *1-2-2-6* above) where there is no contradictions *a prior*. As $\tau(N)$ is large and also $[\tau(N/4)/2]$, it follows the contradiction with $Q(N) \leq 1$ and the hypothesis $(\mu_1, \mu_2) \neq 1$ is false.

** *2-* We suppose that $(\mu_1, \mu_2) = 1$.

We recall that $rad(c) = Y > rad^{1.63/1.37}(a)$, $\delta + 1 = Y$, $rad(a) = r.rad(\delta), (r, rad(\delta)) = 1, \delta = \mu_2 rad(\delta)$ and $r\mu_1 = \delta^2 + 3X$, it follows:

$$U(\delta) = \delta^2 + 3\delta + 3 - r\mu_1 = 0 \tag{74}$$

** *2-1-* We suppose $3|(3 - r\mu_1)$ and $3^2 \nmid (3 - r\mu_1)$, then we use the Eisenstein criterion [6] to the polynomial $U(\delta)$ given by the equation (74), and the contradiction.

** *2-2-* We suppose $3|(3 - r\mu_1)$ and $3^2|(3 - r\mu_1)$. From $3|(3 - r\mu_1) \Longrightarrow 3|r\mu_1 \Longrightarrow 3|r \,\text{or}\, 3|\mu_1$.

- If $3|r \Longrightarrow (3, rad\delta) = 1 \Longrightarrow 3 \nmid \delta$. Then the contradiction with $3|\delta^2$ by the equation (74).
- If $3|\mu_1 \Longrightarrow 3 \nmid \mu_2 \Longrightarrow 3 \nmid \delta$, it follows the contradiction with $3|\delta^2$ by the equation (74).

** *2-3-* We suppose $3 \nmid (3 - r\mu_1) \Longrightarrow 3 \nmid r\mu_1 \Longrightarrow 3 \nmid r$ and $3 \nmid \mu_1$. From the equation (74), $U(\delta) = 0 \Longrightarrow r\mu_1 \equiv \delta^2 (\text{mod} 3)$, as δ^2 is a square then $\delta^2 \equiv 1 (\text{mod} 3) \Longrightarrow r\mu_1 \equiv 1 (\text{mod} 3)$, but this result is not all verified. Then the contradiction.

It follows that the case $\mu_a > rad^{2.26}(a) \Rightarrow a > rad^{3.26}(a)$ and $c = rad^3(c)$ is impossible.

II'-3-2-2- We consider the case $\mu_c = rad^2(c) \Longrightarrow c = rad^3(c)$ and $c = a + b$. Then, we obtain that $Y = rad(c)$ is a solution in positive integers of the equation:

$$Y^3 + 1 = \bar{c} \tag{75}$$

with $\bar{c} = a + b + 1 = c + 1 \Longrightarrow (\bar{c}, c) = 1$. We obtain the same result as of the case **I-3-2-1-** studied above considering $rad(\bar{c}) > rad^{\frac{1.63}{1.37}}(c)$.

II'-3-2-3- We suppose $\mu_a > rad^{2.26}(a) \Rightarrow a > rad^{3.26}(a)$ and c large and $\mu_c < rad^2(c)$, we consider $c = a + b, b \geq 1$. Then $a = rad^3(a) + h, h > 0$, h a positive integer and we can write $c + l = rad^3(c)$, $l > 0$. As $rad(c) >$

$rad^{\frac{1.63}{1.37}}(a) \Longrightarrow rad(c) > rad(a) \Longrightarrow h + l + b = m > 0$, it follows:
(76)
$rad^3(c) - l = rad^3(a) + h + b > 0 \Longrightarrow rad^3(c) - rad^3(a) = h + l + b = m > 0$
We obtain the same result (a contradiction) as of the case **I-3-2-3-** studied above considering $rad(c) > rad^{\frac{1.63}{1.37}}(a)$. Then, this case is to reject.

Then the cases $\mu_c \leq rad^2(c)$ and $a > rad^{3.26}(a)$ are impossible.

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