Schrödinger–Robertson uncertainty relation which depends on the quantum phase transition

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Abstract

We derive the Schrödinger–Robertson uncertainty relation which depends on the quantum phase transition. Our general uncertainty relation asserts, in different times t and t', a fundamental limit to the precision with which certain pairs of physical properties of a particle known as complementary variables, such as its position at time t ($\hat{x}(t)$) and momentum at time t' ($\hat{p}(t')$), can be known. It turns out that the uncertainty relation is valid for different times t and t'. Additionally, it turns out that the formula is natural from the understandable upper limit in the Bloch sphere, in qubits handling, and the meaningful lower limit (exactly zero). We hope the new formula is useful for analyzing for several systems in condensed matter and certain atomic nuclei in which such phase transitions can be observed.

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I. INTRODUCTION

Quantum mechanics (cf. [1–7]) gives accurate and at-times-remarkably accurate numerical predictions and much experimental data has fit to quantum predictions for long time. In quantum mechanics, the uncertainty principle is any of the variety of mathematical inequalities asserting a fundamental limit to the precision with which certain pairs of physical properties of a particle known as complementary variables, such as its position $\hat{x}$ and momentum $\hat{p}$, can be known simultaneously. For instance, in 1927, Werner Heisenberg stated that the more precisely the position of some particle is determined, the less precisely its momentum can be known, and vice versa [8]. The formal inequality relating to the standard deviation of position σ_x and the standard deviation of momentum σ_p was derived by Earle Hesse Kennard [9] later that year and by Hermann Weyl [10] in 1928.

The Schrödinger–Robertson uncertainty relation which is free from the quantum phase transition is as follows:

$$\sigma_A\sigma_B \geq \sqrt{\left(\frac{1}{2}\langle\Psi|\{\hat{A},\hat{B}\}|\Psi\rangle - \langle\Psi|\hat{A}|\Psi\rangle\langle\Psi|\hat{B}|\Psi\rangle\right)^2 + \left(\frac{1}{2i}\langle\Psi|[\hat{A},\hat{B}]|\Psi\rangle\right)^2}, \tag{1}$$

where $|\Psi\rangle$ is a quantum state, σ_A and σ_B are respectively the standard deviations of the Hermitian operators $\hat{A}$ and $\hat{B}$. For a pair of operators $\hat{A}$ and $\hat{B}$, we may define their commutator as $[\hat{A},\hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$ and their anticommutator as $\{\hat{A},\hat{B}\} = \hat{A}\hat{B} + \hat{B}\hat{A}$.

Maccone and Pati discuss stronger uncertainty relations for all incompatible observables [11]. Quantum dynamics of simultaneously measured non-commuting observables is discussed [12]. Dynamics of a qubit while simultaneously monitoring its relaxation and dephasing are also discussed [13]

The upper limit of the Schrödinger–Robertson uncertainty relation which is free from the quantum phase transition in a two-level system (e.g., electron spin, photon polarizations, and so on) is proposed in [14]. This is certified by the Bloch sphere when we would measure $\hat{\sigma_x}$ and $\hat{\sigma_y}$.

What is the motivation behind this work to be discussed in this paper? Concretely speaking, we want to know more useful and precise uncertainty relation. For example, can we derive an uncertainty relation for quantum phase transitions? Is the uncertainty relation valid for different times t and t'? A quantum phase transition is explained as follows [15]: A phase transition occurs at absolute zero temperature when some parameter such as a magnetic field or pressure is changed. In contrast to ordinary phase transitions, which are associated with thermal fluctuations, quantum phase transitions are associated with quantum fluctuations. There are several systems in condensed matter and certain atomic nuclei in which such phase transitions can be observed. We can take into account the quantum phase transition for the Schrödinger–Robertson uncertainty relation, deriving a new formula. It says that the uncertainty relation is valid for different times t and t'. Moreover, we discuss the fact that the new formula is natural from the understandable upper limit (the Bloch sphere) and the meaningful lower limit (exactly zero) by virtue of the convex argumentation.

The Schrödinger–Robertson uncertainty relation which depends on the quantum phase transition is as follows:

$$\sigma_A(t)\sigma_B(t') \geq \sqrt{\left(\frac{1}{2}\langle\Psi(t)|\{\hat{A},\hat{B}\}|\Psi(t')\rangle - \langle\Psi(t)|\hat{A}|\Psi(t)\rangle\langle\Psi(t')|\hat{B}|\Psi(t')\rangle e^{i\pi(\theta(t')-\theta(t))}\right)^2 + \left(\frac{1}{2i}\langle\Psi(t)|[\hat{A},\hat{B}]|\Psi(t')\rangle\right)^2}. \tag{2}$$

The Schrödinger–Robertson uncertainty relation which is free from the quantum phase transition (1) is a part of our general formula (2) where $t = t'$. It turns out that the uncertainty relation is valid for different times t and t'. Here, we introduce the notation t which relates to the quantum phase transition. The quantum phase transition term between a quantum state $|\Psi(t)\rangle = e^{i\pi\theta(t)}|\Psi\rangle$ and another quantum state $|\Psi(t')\rangle = e^{i\pi\theta(t')}|\Psi\rangle$ is given by $\langle\Psi(t)|\Psi(t')\rangle = e^{i\pi\left(\theta(t')-\theta(t)\right)}$.

In this paper, we derive the Schrödinger–Robertson uncertainty relation which depends on the quantum phase transition. Our general uncertainty relation asserts, in different times t and t', a fundamental limit to the precision with which certain pairs of physical properties of a particle known as complementary variables, such as its position at time t ($\hat{x}(t)$) and momentum at time t' ($\hat{p}(t')$), can be known. It turns out that the uncertainty relation is valid for different times t and t'. Additionally, it turns out that the formula is natural from the understandable upper limit in the Bloch sphere, in qubits handling, and the meaningful lower limit (exactly zero). We hope the new formula is useful for analyzing for several systems in condensed matter and certain atomic nuclei in which such phase transitions can be observed.

II. DERIVATION OF THE SCHRÖDINGER–ROBERTSON UNCERTAINTY RELATION WHICH DEPENDS ON THE QUANTUM PHASE TRANSITION

In this section, we derive the Schrödinger–Robertson uncertainty relation which depends on the quantum phase transition. Here, we introduce the notation t which relates to the quantum phase transition. The quantum phase transition term between a quantum state $|\Psi(t)\rangle = e^{i\pi\theta(t)}|\Psi\rangle$ and another quantum state $|\Psi(t')\rangle = e^{i\pi\theta(t')}|\Psi\rangle$ is given by $\langle\Psi(t)|\Psi(t')\rangle = e^{i\pi\left(\theta(t')-\theta(t)\right)}$.

Parts of this derivation shown here incorporate and build off those shown in Robertson [16], Schrödinger [17], and standard textbooks such as Griffiths [18]. As for the derivation of the Schrödinger–Robertson uncertainty relation which depends on the quantum phase transition, the main point is the Cauchy-Schwarz inequality [19] as shown below:

For any Hermitian operator $\hat{A}$, based upon the definition of variance, we have

$$\sigma_A^2(t) = \langle(\hat{A} - \langle\hat{A}\rangle(t))\Psi(t)|(\hat{A} - \langle\hat{A}\rangle(t))\Psi(t)\rangle, \tag{3}$$

where $\langle\hat{A}\rangle(t) = \langle\Psi(t)|\hat{A}|\Psi(t)\rangle$. We let $|f(t)\rangle = |(\hat{A} - \langle\hat{A}\rangle(t))\Psi(t)\rangle$ and thus

$$\sigma_A^2(t) = \langle f(t)|f(t)\rangle. \tag{4}$$

Similarly, for any other Hermitian operator $\hat{B}$ in the state $|\Psi(t')\rangle$

$$\sigma_B^2(t') = \langle(\hat{B} - \langle\hat{B}\rangle(t'))\Psi(t')|(\hat{B} - \langle\hat{B}\rangle(t'))\Psi(t')\rangle = \langle g(t')|g(t')\rangle, \tag{5}$$

for $|g(t')\rangle = |(\hat{B} - \langle\hat{B}\rangle(t'))\Psi(t')\rangle$ and $\langle\hat{B}\rangle(t') = \langle\Psi(t')|\hat{B}|\Psi(t')\rangle$. Thus, the product of the two variances can be expressed as

$$\sigma_A^2(t)\sigma_B^2(t') = \langle f(t)|f(t)\rangle\langle g(t')|g(t')\rangle. \tag{6}$$

In order to relate the two vectors $|f(t)\rangle$ and $|g(t')\rangle$ with each other, we use the Cauchy-Schwarz inequality [19] which is defined as

$$\langle f(t)|f(t)\rangle\langle g(t')|g(t')\rangle \geq |\langle f(t)|g(t')\rangle|^2, \tag{7}$$

and thus Eq. (6) can be written as

$$\sigma_A^2(t)\sigma_B^2(t') \geq |\langle f(t)|g(t')\rangle|^2. \tag{8}$$

Since $\langle f(t)|g(t')\rangle$ is generally a complex number, we use the fact that the modulus squared of any complex number z is defined as $|z|^2 = zz^*$, where z^* is the complex conjugate of z. The modulus squared can also be expressed as

$$|z|^2 = (\mathrm{Re}(z))^2 + (\mathrm{Im}(z))^2 = \left(\frac{z+z^*}{2}\right)^2 + \left(\frac{z-z^*}{2i}\right)^2. \tag{9}$$

We let $z = \langle f(t)|g(t')\rangle$ and $z^* = \langle g(t')|f(t)\rangle$ and substitute these into the equation above in giving

$$|\langle f(t)|g(t')\rangle|^2 = \left(\frac{\langle f(t)|g(t')\rangle + \langle g(t')|f(t)\rangle}{2}\right)^2 + \left(\frac{\langle f(t)|g(t')\rangle - \langle g(t')|f(t)\rangle}{2i}\right)^2. \tag{10}$$

The inner product $\langle f(t)|g(t')\rangle$ is written out explicitly as

$$\langle f(t)|g(t')\rangle = \langle(\hat{A} - \langle\hat{A}\rangle(t))\Psi(t)|(\hat{B} - \langle\hat{B}\rangle(t'))\Psi(t')\rangle, \tag{11}$$

and using the fact that $\hat{A}$ and $\hat{B}$ are Hermitian operators, we find

$$\begin{aligned}\langle f(t)|g(t')\rangle &= \langle\Psi(t)|(\hat{A} - \langle\hat{A}\rangle(t))(\hat{B} - \langle\hat{B}\rangle(t'))\Psi(t')\rangle \\ &= \langle\Psi(t)|(\hat{A}\hat{B} - \hat{A}\langle\hat{B}\rangle(t') - \hat{B}\langle\hat{A}\rangle(t) + \langle\hat{A}\rangle(t)\langle\hat{B}\rangle(t'))\Psi(t')\rangle \\ &= \langle\Psi(t)|\hat{A}\hat{B}\Psi(t')\rangle - \langle\Psi(t)|\hat{A}\langle\hat{B}\rangle(t')\Psi(t')\rangle - \langle\Psi(t)|\hat{B}\langle\hat{A}\rangle(t)\Psi(t')\rangle + \langle\Psi(t)|\langle\hat{A}\rangle(t)\langle\hat{B}\rangle(t')\Psi(t')\rangle \\ &= \langle\Psi(t)|\hat{A}\hat{B}\Psi(t')\rangle - \langle\hat{A}\rangle(t)\langle\hat{B}\rangle(t')\langle\Psi(t)|\Psi(t')\rangle - \langle\hat{A}\rangle(t)\langle\hat{B}\rangle(t')\langle\Psi(t)|\Psi(t')\rangle + \langle\hat{A}\rangle(t)\langle\hat{B}\rangle(t')\langle\Psi(t)|\Psi(t')\rangle \\ &= \langle\Psi(t)|\hat{A}\hat{B}\Psi(t')\rangle - \langle\hat{A}\rangle(t)\langle\hat{B}\rangle(t')\langle\Psi(t)|\Psi(t')\rangle.\end{aligned} \tag{12}$$

Similarly, it can be shown that $\langle g(t')|f(t)\rangle = \langle\Psi(t)|\hat{B}\hat{A}\Psi(t')\rangle - \langle\hat{A}\rangle(t)\langle\hat{B}\rangle(t')\langle\Psi(t)|\Psi(t')\rangle$. For a pair of operators $\hat{A}$ and $\hat{B}$, we may define their commutator as $[\hat{A},\hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$. Thus we have

$$\begin{aligned}&\langle f(t)|g(t')\rangle - \langle g(t')|f(t)\rangle = \langle\Psi(t)|\hat{A}\hat{B}\Psi(t')\rangle - \langle\hat{A}\rangle(t)\langle\hat{B}\rangle(t')\langle\Psi(t)|\Psi(t')\rangle\\ &-\langle\Psi(t)|\hat{B}\hat{A}\Psi(t')\rangle + \langle\hat{A}\rangle(t)\langle\hat{B}\rangle(t')\langle\Psi(t)|\Psi(t')\rangle\\ &= \langle\Psi(t)|[\hat{A},\hat{B}]|\Psi(t')\rangle\end{aligned} \tag{13}$$

and

$$\begin{aligned}&\langle f(t)|g(t')\rangle + \langle g(t')|f(t)\rangle = \langle\Psi(t)|\hat{A}\hat{B}\Psi(t')\rangle - \langle\hat{A}\rangle(t)\langle\hat{B}\rangle(t')\langle\Psi(t)|\Psi(t')\rangle\\ &+\langle\Psi(t)|\hat{B}\hat{A}\Psi(t')\rangle - \langle\hat{A}\rangle(t)\langle\hat{B}\rangle(t')\langle\Psi(t)|\Psi(t')\rangle\\ &= \langle\Psi(t)|\{\hat{A},\hat{B}\}|\Psi(t')\rangle - 2\langle\hat{A}\rangle(t)\langle\hat{B}\rangle(t')\langle\Psi(t)|\Psi(t')\rangle,\end{aligned} \tag{14}$$

where we introduce the anticommutator $\{\hat{A},\hat{B}\} = \hat{A}\hat{B}+\hat{B}\hat{A}$. We now substitute the above two equations into Eq. (10) in giving

$$|\langle f(t)|g(t')\rangle|^2 = \left(\frac{1}{2}\langle\Psi(t)|\{\hat{A},\hat{B}\}|\Psi(t')\rangle - \langle\hat{A}\rangle(t)\langle\hat{B}\rangle(t')\langle\Psi(t)|\Psi(t')\rangle\right)^2 + \left(\frac{1}{2i}\langle\Psi(t)|[\hat{A},\hat{B}]|\Psi(t')\rangle\right)^2. \tag{15}$$

Substituting the above into Eq. (8), we get the Schrödinger–Robertson uncertainty relation which depends on the quantum phase transition as follows:

$$\sigma_A(t)\sigma_B(t') \geq \sqrt{\left(\frac{1}{2}\langle\Psi(t)|\{\hat{A},\hat{B}\}|\Psi(t')\rangle - \langle\hat{A}\rangle(t)\langle\hat{B}\rangle(t')e^{i\pi(\theta(t')-\theta(t))}\right)^2 + \left(\frac{1}{2i}\langle\Psi(t)|[\hat{A},\hat{B}]|\Psi(t')\rangle\right)^2}, \tag{16}$$

where $\langle\Psi(t)|\Psi(t')\rangle = e^{i\pi\left(\theta(t')-\theta(t)\right)}$. The Schrödinger–Robertson uncertainty relation which is free from the quantum phase transition is a part of our general formula where $t = t'$, i.e.,

$$\sigma_A\sigma_B \geq \sqrt{\left(\frac{1}{2}\langle\Psi|\{\hat{A},\hat{B}\}|\Psi\rangle - \langle\hat{A}\rangle\langle\hat{B}\rangle\right)^2 + \left(\frac{1}{2i}\langle\Psi|[\hat{A},\hat{B}]|\Psi\rangle\right)^2}. \tag{17}$$

III. UPPER LIMIT OF THE SCHRÖDINGER–ROBERTSON UNCERTAINTY RELATION WHICH DEPENDS ON THE QUANTUM PHASE TRANSITION

The upper limit of the Schrödinger–Robertson uncertainty relation which depends on the quantum phase transition can be derived by the Schrödinger–Robertson uncertainty relation which is free from the quantum phase transition, because of the convex argumentation.

In this section, we discuss the fact that the Bloch sphere imposes the upper limit of the Schrödinger–Robertson uncertainty relation which depends on the quantum phase transition. We derive the Schrödinger–Robertson uncertainty relation which is free from the quantum phase transition by using the Bloch sphere in the specific case. Let σ_X^2 be the variance of $\hat{X}$, i.e., $\langle\hat{X}^2\rangle - \langle\hat{X}\rangle^2$.

The Schrödinger–Robertson uncertainty relation which is free from the quantum phase transition is described before (17). We derive a specific example [14] that the Bloch sphere imposes the upper limit of the Schrödinger–Robertson uncertainty relation which is free from the quantum phase transition. Let $\hat{A} = \hat{\sigma_x}$ and $\hat{B} = \hat{\sigma_y}$ in giving

$$\sigma_{\sigma_x}\sigma_{\sigma_y} \geq \sqrt{\left(\frac{1}{2}\langle\Psi|\{\hat{\sigma_x},\hat{\sigma_y}\}|\Psi\rangle - \langle\hat{\sigma_x}\rangle\langle\hat{\sigma_y}\rangle\right)^2 + \left(\frac{1}{2i}\langle\Psi|[\hat{\sigma_x},\hat{\sigma_y}]|\Psi\rangle\right)^2}. \tag{18}$$

Thus, as $\langle\Psi|\{\hat{\sigma_x},\hat{\sigma_y}\}|\Psi\rangle = 0$, we have the following:

$$\sigma_{\sigma_x}\sigma_{\sigma_y} \geq \sqrt{\langle\hat{\sigma_x}\rangle^2\langle\hat{\sigma_y}\rangle^2 + \left(\frac{1}{2i}\langle\Psi|[\hat{\sigma_x},\hat{\sigma_y}]|\Psi\rangle\right)^2}. \tag{19}$$

The Schrödinger–Robertson uncertainty relation which is free from the quantum phase transition is derived from the Bloch sphere as shown below:

Statement 1

$$1-\langle\hat{\sigma_x}\rangle^2-\langle\hat{\sigma_y}\rangle^2-\langle\hat{\sigma_z}\rangle^2\geq 0\Rightarrow \sigma_{\sigma_x}\sigma_{\sigma_y}\geq\sqrt{\langle\hat{\sigma_x}\rangle^2\langle\hat{\sigma_y}\rangle^2+\left(\frac{1}{2i}\langle[\hat{\sigma_x},\hat{\sigma_y}]\rangle\right)^2}. \tag{20}$$

Proof: By using $1-\langle\hat{\sigma_x}\rangle^2-\langle\hat{\sigma_y}\rangle^2\geq\langle\hat{\sigma_z}\rangle^2$, we have

$$\sigma^2_{\sigma_x}\sigma^2_{\sigma_y}=(1-\langle\hat{\sigma_x}\rangle^2)(1-\langle\hat{\sigma_y}\rangle^2)=1-\langle\hat{\sigma_x}\rangle^2-\langle\hat{\sigma_y}\rangle^2+\langle\hat{\sigma_x}\rangle^2\langle\hat{\sigma_y}\rangle^2\geq\langle\hat{\sigma_z}\rangle^2+\langle\hat{\sigma_x}\rangle^2\langle\hat{\sigma_y}\rangle^2$$
$$=\left(\frac{1}{2i}\langle[\hat{\sigma_x},\hat{\sigma_y}]\rangle\right)^2+\langle\hat{\sigma_x}\rangle^2\langle\hat{\sigma_y}\rangle^2. \tag{21}$$

Thus,

$$\sigma_{\sigma_x}\sigma_{\sigma_y}\geq\sqrt{\langle\hat{\sigma_x}\rangle^2\langle\hat{\sigma_y}\rangle^2+\left(\frac{1}{2i}\langle[\hat{\sigma_x},\hat{\sigma_y}]\rangle\right)^2}. \tag{22}$$

QED

The upper limit of the Schrödinger–Robertson uncertainty relation which is free from the quantum phase transition in a two-level system (e.g., electron spin, photon polarizations, and so on) is derived by [14]. This is certified by the Bloch sphere when we would measure $\hat{\sigma_x}$ and $\hat{\sigma_y}$. Therefore, the Bloch sphere imposes the upper limit of the Schrödinger–Robertson uncertainty relation which is free from the quantum phase transition. It turns out that the Schrödinger–Robertson uncertainty relation which depends on the quantum phase transition is natural from the understandable upper limit (the Bloch sphere) by virtue of the convex argumentation.

IV. LOWER LIMIT OF THE SCHRÖDINGER–ROBERTSON UNCERTAINTY RELATION WHICH DEPENDS ON THE QUANTUM PHASE TRANSITION

The lower limit of the Schrödinger–Robertson uncertainty relation which depends on the quantum phase transition can be derived by the Schrödinger–Robertson uncertainty relation which is free from the quantum phase transition, because of the convex argumentation.

We suppose that $\hat{A},\hat{B}$ are two Hermitian operators on an N-dimensional unitary space. Let us consider a simultaneous pure eigenstate $|\Psi_i\rangle,(i=1,2,...,N)$, that is, $\langle\Psi_i|\Psi_j\rangle=\delta_{ij}$, for the two Hermitian operators $\hat{A},\hat{B}$ such that $\langle\Psi_i|\hat{A}|\Psi_i\rangle=a_i,\langle\Psi_i|\hat{B}|\Psi_i\rangle=b_i$.

The Schrödinger–Robertson uncertainty relation which depends on the quantum phase transition is as shown in (16). The Schrödinger–Robertson uncertainty relation which is free from the quantum phase transition is as shown in (17).

Statement 2

When $[\hat{A},\hat{B}]=0$, the Schrödinger–Robertson uncertainty relation which is free from the quantum phase transition becomes

$$\sigma_A\sigma_B\geq\langle\hat{A}\hat{B}\rangle-\langle\hat{A}\rangle\langle\hat{B}\rangle, \tag{23}$$

and the lower bound is zero.

Proof: We consider the Schrödinger–Robertson uncertainty relation which is free from the quantum phase transition in the case where $[\hat{A},\hat{B}]=0$

$$\sigma_A\sigma_B\geq\sqrt{\left(\frac{1}{2}\langle\{\hat{A},\hat{B}\}\rangle-\langle\hat{A}\rangle\langle\hat{B}\rangle\right)^2}. \tag{24}$$

Thus, we have

$$\sigma_A\sigma_B\geq\langle\hat{A}\hat{B}\rangle-\langle\hat{A}\rangle\langle\hat{B}\rangle. \tag{25}$$

On the other hand, we have

$$\langle\Psi_i|\hat{A}\hat{B}|\Psi_i\rangle=a_ib_i,$$
$$\langle\Psi_i|\hat{A}|\Psi_i\rangle\langle\Psi_i|\hat{B}|\Psi_i\rangle=a_ib_i, \tag{26}$$

where $[\hat{A},\hat{B}]=0$ and a_i,b_i are respectively eigenvalues of the two Hermitian operators $\hat{A}$ and $\hat{B}$.

QED

We show that the lower bound of the Schrödinger–Robertson uncertainty relation which is free from the quantum phase transition is exactly zero. Therefore, the Schrödinger–Robertson uncertainty relation which depends on the quantum phase transition is also exactly zero by virtue of the convex argumentation. It turns out that the Schrödinger–Robertson uncertainty relation which depends on the quantum phase transition says a precise measurement on commuting observables, symmetric measurement [20], is possible.

We hope the new formula is useful for analyzing for several systems in condensed matter and certain atomic nuclei in which such phase transitions can be observed.

V. CONCLUSIONS

In conclusions, we have derived the Schrödinger–Robertson uncertainty relation which depends on the quantum phase transition. Our general uncertainty relation has asserted, in different times t and t', a fundamental limit to the precision with which certain pairs of physical properties of a particle known as complementary variables, such as its position at time t ($\hat{x}(t)$) and momentum at time t' ($\hat{p}(t')$), can be known. It has turned out that the uncertainty relation is valid for different times t and t'. Additionally, it has turned out that the formula is natural from the understandable upper limit in the Bloch sphere, in qubits handling, and the meaningful lower limit (exactly zero). We have hoped the new formula is useful for analyzing for several systems in condensed matter and certain atomic nuclei in which such phase transitions can be observed.

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Ethical Approval

The authors are in an applicable thought to ethical approval.

Competing Interests

The authors state that there is no conflict of interest.

Author Contributions

Koji Nagata, Do Ngoc Diep, and Tadao Nakamura wrote and read the manuscript.

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No data associated in the manuscript.

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