Theoretically organized algorithms based on orthogonal states

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Abstract

We expand Deutsch's algorithm for determining all the mappings of a function by using four orthogonal states. Using this, we propose a parallel computation for all of the combinations of values in variables of a logical function by using sixteen orthogonal states. As an application of our algorithm, we demonstrate two typical arithmetic calculations in the binary system. We study an efficiency for operating a full adder/half adder by quantum-gated computing. The two typical arithmetic calculations are $(1+1)$ and $(2+3)$. The typical arithmetic calculation $(2+3)$ is faster than that of its classical apparatus which would require $2^{16} = 4096$ steps when we introduce the full adder operation. Another typical arithmetic calculation $(1+1)$ is faster than that of its classical apparatus which would require $2^8 = 256$ steps when we introduce only the half adder operation.

Keywords: Quantum algorithms, Quantum computation, Boolean algebra, Parallel computing

I. INTRODUCTION

Quantum mechanics (cf. [1–7]) is an important physical theory in order to explain the quantum behaviors of the nature. Between the articles of research for constructing theoretical quantum algorithms [8] it may be mentioned as follows. In 1985, the Deutsch algorithm was introduced and constructed for the function property problem [9–11]. In 1993, the Bernstein–Vazirani algorithm was proposed for identifying linear functions [12, 13]. Generalization of the Bernstein–Vazirani algorithm beyond qubit systems is reported [14]. In 1994, Simon's algorithm [15] and Shor's algorithm [16] were discussed for period finding of periodic functions. In 1996, Grover [17] provided an algorithm for unordered object finding and the motivation for exploring the computational possibilities offered by quantum mechanics.

A simple algorithm for complete factorization of an N-partite pure quantum state is proposed by Mehendale and Joag [18]. Fujikawa, Oh, and Umetsu discuss a classical limit of Grover's algorithm induced by dephasing: coherence versus entanglement [19]. Quantum dialogue protocol based on Grover's search algorithms is presented by Yin, He, and Fan [20]. Efficient quantum arithmetic operation circuits for quantum image processing are discussed by Li *et al.* [21]. Gidney discusses halving the cost of quantum addition [22]. Li *et al.* discuss the circuit design and optimization of quantum multiplier and divider [23].

Some related references may be included on the high-dimensional case. Yan and Gao discuss Perfect NOT and conjugate transformations which are about the perfect NOT gate in d-dimensions [24]. Liu *et al.* discuss general scheme for superdense coding between multiparties [25]. Deconstructing dense coding is discussed by Mermin [26]. In high dimension, some of the operations are more complicated than that in the qubit case.

Some relations between a boolean algebra and quantum computing are discussed and proposed by Nagata and Nakamura. They show all the boolean functions are set into the quantum computer just like the electronic computer. This fact means that all performances in logic of computing and control of itself are available even in quantum computers. Therefore, we could design any quantum-gated computer using the traditional design ways in logic of existing electronic computers [27].

Further they prove that the quantum computer can operate just like the electronic computer fundamentally through the operation of addition of two n-digit numbers. Therefore, the quantum computer can solve all the four basic operations of arithmetic, addition, subtraction, multiplication, and division. Further it can be said that this quantum computer naturally operates not only arithmetic but also logic in terms of boolean logic [28]. As a result, the theory proposed by Refs. [27, 28] can build a very true quantum-gated computer that is driven and operated by all software (all programs) used on existing electronic computers. Generalization of Deutsch's algorithm is reported [29]. Quantum algorithm for evaluating two of logical functions simultaneously is proposed [30]. A quantum algorithm for a FULL adder operation based on registers of the CPU in a quantum-gated computer is discussed [31]. Computational complexity in high-dimensional quantum computing is also discussed [32].

However, in the theory presented in Refs. [27–29], all the quantum states are not completely orthogonal to each other. Therefore, we have some error probability when we distinguish the quantum states [33, 34]. Nevertheless, we are able to construct our theory by using orthogonal states.

In this article, we completely and simply expand Deutsch's algorithm for determining all the mappings of a function by using four orthogonal states. Using this, we propose a direct and simple method for a parallel computation for all of the combinations of values in variables of a logical function by using sixteen orthogonal states. As an application of our algorithm, we demonstrate two typical arithmetic calculations in the binary system. We study an efficiency for operating a full adder/half adder by quantum-gated computing. The two typical arithmetic calculations are $(1+1)$ and $(2+3)$. The typical arithmetic calculation $(2+3)$ is faster than that of its classical apparatus which would require $2^{16} = 4096$ steps when we introduce the full adder operation. Another typical arithmetic calculation $(1+1)$ is faster than that of its classical apparatus which would require $2^8 = 256$ steps when we introduce only the half adder operation.

II. EXPANSION OF DEUTSCH'S ALGORITHM BASED ON ORTHOGONAL STATES

Deutsch's algorithm determines if the given function $f : \mathbb{Z}_2 \to \mathbb{Z}_2$ is constant or balanced. The function is called to be constant if $f(0) = f(1)$. The function is called to be balanced if $f(0) \neq f(1)$. We expand Deutsch's algorithm. Deutsch's algorithm expanded determines all the mappings of the given function. We can determine simultaneously the following mappings:

$$f(0) = ?, f(1) = ?. \tag{1}$$

1. Basic structure of quantum computing

Quantum superposition and quantum phase factor are fundamental features of many quantum algorithms. Both of them are necessary. They allow quantum computers to evaluate simultaneously the mappings of a function $f(x)$ for many different x. Suppose

$$f:\{0,1\}\to\{0,1\} \tag{2}$$

is a function with a one-bit domain and range. A convenient way of computing the function on a quantum computer is of considering a two-qubit quantum computer that starts with the state $|x,y\rangle$, where x and y are variables used in mapping f. The abbreviation $|x,y\rangle$ stands for $|x\rangle\otimes|y\rangle$.

It is possible to transform the state $|x,y\rangle$ into

$$|x,y\oplus f(x)\rangle, \tag{3}$$

by applying the quantum oracle, where $\oplus$ indicates addition modulo 2. We denote the transformation U_f defined by the map

$$U_f|x,y\rangle=|x,y\oplus f(x)\rangle. \tag{4}$$

2. Deutsch's algorithm

Let us review Deutsch's formula as follows:

$$\begin{aligned}&U_f|0\rangle(|0\rangle-|1\rangle)/\sqrt{2}=|0\rangle(|0\oplus f(0)\rangle-|1\oplus f(0)\rangle)/\sqrt{2}\\&=\begin{cases}(-1)^{f(0)}|0\rangle(|0\rangle-|1\rangle)/\sqrt{2} & \text{if}\quad f(0)=0,\\(-1)^{f(0)}|0\rangle(|0\rangle-|1\rangle)/\sqrt{2} & \text{if}\quad f(0)=1.\end{cases}\end{aligned} \tag{5}$$

$$\begin{aligned}&U_f|1\rangle(|0\rangle-|1\rangle)/\sqrt{2}=|1\rangle(|0\oplus f(1)\rangle-|1\oplus f(1)\rangle)/\sqrt{2}\\&=\begin{cases}(-1)^{f(1)}|1\rangle(|0\rangle-|1\rangle)/\sqrt{2} & \text{if}\quad f(1)=0,\\(-1)^{f(1)}|1\rangle(|0\rangle-|1\rangle)/\sqrt{2} & \text{if}\quad f(1)=1.\end{cases}\end{aligned} \tag{6}$$

This is the phase kickback formation.

Let us introduce the Bloch sphere. We consider a quantum state lying in the x-axis and a quantum state lying in the z-axis. Deutsch's formula does not use a quantum state lying in y-axis. $f(0)$ and $f(1)$ appear in the global phase factor, but we cannot obtain both of them at the same time.

We define the following notations:

$$|-\rangle_y=\frac{|0\rangle-i|1\rangle}{\sqrt{2}},|+\rangle_y=\frac{|0\rangle+i|1\rangle}{\sqrt{2}},|-\rangle_x=\frac{|0\rangle-|1\rangle}{\sqrt{2}},|+\rangle_x=\frac{|0\rangle+|1\rangle}{\sqrt{2}}. \tag{7}$$

In fact, we may define the input state as (8).

$$|\psi_0\rangle_d=\frac{1}{\sqrt{2}}|0\rangle|-\rangle_x+\frac{1}{\sqrt{2}}|1\rangle|-\rangle_x=|+\rangle_x|-\rangle_x,\quad {}_d\langle\psi_0|\psi_0\rangle_d=1. \tag{8}$$

Applying $U_{f_i},(i=0,1,2,3)$ to $|\psi_0\rangle_d$, $U_{f_i}|\psi_0\rangle_d=|\psi_1\rangle_{id}$, therefore leaves us with one of four cases:

$$\begin{aligned}U_{f_0}|\psi_0\rangle_d=|\psi_1\rangle_{0d}&=\frac{1}{\sqrt{2}}|0\rangle|-\rangle_x+\frac{1}{\sqrt{2}}|1\rangle|-\rangle_x=|+\rangle_x|-\rangle_x\quad\text{iff}\quad f_0(0)=0,f_0(1)=0,\\U_{f_1}|\psi_0\rangle_d=|\psi_1\rangle_{1d}&=\frac{1}{\sqrt{2}}|0\rangle|-\rangle_x-\frac{1}{\sqrt{2}}|1\rangle|-\rangle_x=|-\rangle_x|-\rangle_x\quad\text{iff}\quad f_1(0)=0,f_1(1)=1,\\U_{f_2}|\psi_0\rangle_d=|\psi_1\rangle_{2d}&=-\frac{1}{\sqrt{2}}|0\rangle|-\rangle_x+\frac{1}{\sqrt{2}}|1\rangle|-\rangle_x=-|-\rangle_x|-\rangle_x\quad\text{iff}\quad f_2(0)=1,f_2(1)=0,\\U_{f_3}|\psi_0\rangle_d=|\psi_1\rangle_{3d}&=-\frac{1}{\sqrt{2}}|0\rangle|-\rangle_x-\frac{1}{\sqrt{2}}|1\rangle|-\rangle_x=-|+\rangle_x|-\rangle_x\quad\text{iff}\quad f_3(0)=1,f_3(1)=1.\end{aligned} \tag{9}$$

If we have (9), we do not obtain simultaneously both $f(0)$ and $f(1)$ by measuring the single output state.

By measuring $|\psi_1\rangle_{id}$, we cannot determine simultaneously all the two mappings of $f_i(x)$ for all x. But, we can determine if the given function is constant or balanced. This is very interesting indeed: the quantum algorithm gives us the ability to determine a property of $f_i(x)$. This is faster than a classical apparatus, which would require at least 2 evaluations.

3. Improvement of Deutsch's algorithm

We can improve Deutsch's algorithm using a quantum state lying in the xy-plane. In what follows, we consider the Bloch sphere, especially, we consider a quantum state lying in the xy-plane. From Deutsch's formula and the mapping U_f, we arrive the following formulas:

$$U_f|0\rangle(\cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle) = |0\rangle(\cos\frac{\theta}{2}|0\oplus f(0)\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\oplus f(0)\rangle)$$
$$= \begin{cases} |0\rangle(\cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle) & \text{if } f(0)=0, \\ |0\rangle(\cos\frac{\theta}{2}|1\rangle + e^{i\phi}\sin\frac{\theta}{2}|0\rangle) & \text{if } f(0)=1. \end{cases} \tag{10}$$

$$U_f|1\rangle(\cos\frac{\theta'}{2}|0\rangle + e^{i\phi'}\sin\frac{\theta'}{2}|1\rangle) = |1\rangle(\cos\frac{\theta'}{2}|0\oplus f(1)\rangle + e^{i\phi'}\sin\frac{\theta'}{2}|1\oplus f(1)\rangle)$$
$$= \begin{cases} |1\rangle(\cos\frac{\theta'}{2}|0\rangle + e^{i\phi'}\sin\frac{\theta'}{2}|1\rangle) & \text{if } f(1)=0, \\ |1\rangle(\cos\frac{\theta'}{2}|1\rangle + e^{i\phi'}\sin\frac{\theta'}{2}|0\rangle) & \text{if } f(1)=1. \end{cases} \tag{11}$$

This is enough to realize our main goal, but, to simplify, we suppose a quantum state lying in just the y-axis. So let (θ,ϕ) be $(\pi/2,\pi/2)$ and let (θ',ϕ') be $(\pi/2,\pi/2)$ in giving

$$U_f|0\rangle(|0\rangle + i|1\rangle)/\sqrt{2} = \begin{cases} (i)^{f(0)}|0\rangle(|0\rangle + i|1\rangle)/\sqrt{2} & \text{if } f(0)=0, \\ (i)^{f(0)}|0\rangle(|0\rangle - i|1\rangle)/\sqrt{2} & \text{if } f(0)=1. \end{cases} \tag{12}$$

$$U_f|1\rangle(|0\rangle + i|1\rangle)/\sqrt{2} = \begin{cases} (i)^{f(1)}|1\rangle(|0\rangle + i|1\rangle)/\sqrt{2} & \text{if } f(1)=0, \\ (i)^{f(1)}|1\rangle(|0\rangle - i|1\rangle)/\sqrt{2} & \text{if } f(1)=1. \end{cases} \tag{13}$$

We define the input state as (14). Here we use a phase effect, which is a quantum phenomenon. We define the input state as follows, using an imaginary number i:

$$|\psi_0\rangle = \frac{1}{\sqrt{2}}|0\rangle|+\rangle_y + \frac{1}{\sqrt{2}}|1\rangle|+\rangle_y, \langle\psi_0|\psi_0\rangle = 1. \tag{14}$$

Applying $U_{f_i}, (i=0,1,2,3)$ to $|\psi_0\rangle$, $U_{f_i}|\psi_0\rangle = |\psi_1\rangle_i$, therefore leaves us with one of four cases:

$$U_{f_0}|\psi_0\rangle = |\psi_1\rangle_0 = \frac{1}{\sqrt{2}}|0\rangle|+\rangle_y + \frac{1}{\sqrt{2}}|1\rangle|+\rangle_y \quad \text{iff} \quad f_0(0)=0, f_0(1)=0,$$
$$U_{f_1}|\psi_0\rangle = |\psi_1\rangle_1 = \frac{1}{\sqrt{2}}|0\rangle|+\rangle_y + i\frac{1}{\sqrt{2}}|1\rangle|-\rangle_y \quad \text{iff} \quad f_1(0)=0, f_1(1)=1,$$
$$U_{f_2}|\psi_0\rangle = |\psi_1\rangle_2 = i\frac{1}{\sqrt{2}}|0\rangle|-\rangle_y + \frac{1}{\sqrt{2}}|1\rangle|+\rangle_y \quad \text{iff} \quad f_2(0)=1, f_2(1)=0,$$
$$U_{f_3}|\psi_0\rangle = |\psi_1\rangle_3 = i\frac{1}{\sqrt{2}}|0\rangle|-\rangle_y + i\frac{1}{\sqrt{2}}|1\rangle|-\rangle_y \quad \text{iff} \quad f_3(0)=1, f_3(1)=1. \tag{15}$$

If we have (15), we know simultaneously both $f(0)$ and $f(1)$ by measuring the single output state.

Thus, by measuring $|\psi_1\rangle_i$, we may determine simultaneously all the two mappings of $f_i(x)$ for all x. This is very interesting indeed: the quantum algorithm gives us the ability to determine a perfect property of $f_i(x)$, namely, $f_i(x)$ itself. This is faster than a classical apparatus, which would require at least 2^2 evaluations. However, the four states are not orthogonal to each other. Therefore, we have some error probability when we distinguish the four states [33, 34]. Nevertheless, we are able to eliminate the error probability into zero as we show below.

4. Expansion of Deutsch's algorithm based on orthogonal states

We present the expansion of Deutsch's algorithm based on orthogonal states. We propose the following input state:

$$|\psi_0\rangle_d \otimes |\psi_0\rangle = |+\rangle_x|-\rangle_x \otimes |+\rangle_x|+\rangle_y. \tag{16}$$

Applying $U_{f_i}\otimes U_{f_i}, (i=0,1,2,3)$ to $|\psi_0\rangle_d \otimes |\psi_0\rangle$, $U_{f_i}\otimes U_{f_i}|\psi_0\rangle_d \otimes |\psi_0\rangle = |\psi_1\rangle_{id} \otimes |\psi_1\rangle_i$, therefore leaves us with one of four cases:

$$U_{f_0}\otimes U_{f_0}|\psi_0\rangle_d \otimes |\psi_0\rangle = |\psi_1\rangle_{0d} \otimes |\psi_1\rangle_0 = |+\rangle_x|-\rangle_x \otimes (\frac{1}{\sqrt{2}}|0\rangle|+\rangle_y + \frac{1}{\sqrt{2}}|1\rangle|+\rangle_y)$$
$$\text{iff} \quad f_0(0)=0, f_0(1)=0, \tag{17}$$

$$U_{f_1} \otimes U_{f_1} |\psi_0\rangle_d \otimes |\psi_0\rangle = |\psi_1\rangle_{1d} \otimes |\psi_1\rangle_1 = |-\rangle_x |-\rangle_x \otimes (\frac{1}{\sqrt{2}}|0\rangle|+\rangle_y + i\frac{1}{\sqrt{2}}|1\rangle|-\rangle_y)$$
$$\text{iff} \quad f_1(0) = 0, f_1(1) = 1, \tag{18}$$

$$U_{f_2} \otimes U_{f_2} |\psi_0\rangle_d \otimes |\psi_0\rangle = |\psi_1\rangle_{2d} \otimes |\psi_1\rangle_2 = -|-\rangle_x |-\rangle_x \otimes (i\frac{1}{\sqrt{2}}|0\rangle|-\rangle_y + \frac{1}{\sqrt{2}}|1\rangle|+\rangle_y)$$
$$\text{iff} \quad f_2(0) = 1, f_2(1) = 0, \tag{19}$$

$$U_{f_3} \otimes U_{f_3} |\psi_0\rangle_d \otimes |\psi_0\rangle = |\psi_1\rangle_{3d} \otimes |\psi_1\rangle_3 = -|+\rangle_x |-\rangle_x \otimes (i\frac{1}{\sqrt{2}}|0\rangle|-\rangle_y + i\frac{1}{\sqrt{2}}|1\rangle|-\rangle_y)$$
$$\text{iff} \quad f_3(0) = 1, f_3(1) = 1. \tag{20}$$

If we have the relations above, we know simultaneously both $f(0)$ and $f(1)$ by measuring the single output state. The four states are orthogonal to each other. Therefore, we have zero error probability when we distinguish the four states.

Thus, by measuring $|\psi_1\rangle_{id} \otimes |\psi_1\rangle_i$, we may determine simultaneously all the two mappings of $f_i(x)$ for all x. This is very interesting indeed: the quantum algorithm gives us the ability to determine a perfect property of $f_i(x)$, namely, $f_i(x)$ itself. This is faster than a classical apparatus, which would require at least 2^2 evaluations.

Our algorithm is as follows:

- Select a function f_i and do not know any mappings of it, that is,

$$f_i(0) = ?, f_i(1) = ?. \tag{21}$$

- Operate $U_{f_i} \otimes U_{f_i}$ to $|\psi_0\rangle_d \otimes |\psi_0\rangle$ in giving $|\psi_1\rangle_{id} \otimes |\psi_1\rangle_i$.
- From $|\psi_1\rangle_{id} \otimes |\psi_1\rangle_i$, obtain the values of all the mappings concerning the function f_i.

It is very interesting to consider the Deutsch-Jozsa algorithm expanded which uses N qubits, and in this case, we could evaluate the values of the 2^N mappings at the same time.

5. Relation between the atoms (set theory) and Deutsch's algorithm expanded

A	0 1	
$f0$	0 0	0
$f1$	0 1	A
$f2$	1 0	A'
$f3$	1 1	1

A boolean algebra F_1

Let us discuss the relation between the atoms [35] (set theory) and Deutsch's algorithm expanded. This A is a subset which is constructed using the atoms $f1$ and $f2$ that are disjoint one another. For example, newly using fi as an element of a boolean algebra F_1,

$$\begin{aligned} f0 &= 0, \\ f1 &= A, \\ f2 &= A', \\ f3 &= 1. \end{aligned} \tag{22}$$

We can introduce a boolean algebra F_1 as a power set of the atoms. F_1 is based on the value "1" of the one-variable switching functions. An atom is a function including only one "1" as its mapped value, in the two combinations of the values of A for the one-variable function.

Clearly we notice a complete matching between the boolean algebra F_1 and Deutsch's algorithm expanded. In fact we can see that Eqs. (18) and (19) are regarded as the two atoms of the boolean algebra F_1. For example, we notice (18) OR operation with (19) is equal to (20) and all elements are derived from the two atoms.

We see that the relation between set theory based upon atoms and our result in terms of a boolean algebra. The important point is that we obtain all the elements of F_1 by means of a power set of atoms when we get the two atoms. Thus we can say that next our aim is of getting simultaneously (18) and (19). That means we get simultaneously (17)–(20) (all four patterns!). This is now possible as we discuss: We can construct very clearly the following quantum state composed on two orthogonal states:

$$(|\psi_1\rangle_{1d} \otimes |\psi_1\rangle_1) \otimes (|\psi_1\rangle_{2d} \otimes |\psi_1\rangle_2). \tag{23}$$

And we evaluate this quantum state of obtaining all the mappings. Especially, we have a quantum algorithm for evaluating two of logical functions simultaneously [30] and then we have

$$\frac{(|\psi_1\rangle_{1d} \otimes |\psi_1\rangle_1) + i(|\psi_1\rangle_{2d} \otimes |\psi_1\rangle_2)}{\sqrt{2}}. \tag{24}$$

In this case, we evaluate one quantum state of obtaining all the mappings.

III. THEORETICALLY ORGANIZED ALGORITHM FOR QUANTUM COMPUTERS BASED ON ORTHOGONAL STATES

The key of this section is to develop the algorithms of quantum computers toward the ultimate parallel processing on them. The way to do is to find out the very true ultimate parallelism, thinking of the physical quantum phenomena. The algorithm developed here is toward the full uses of the features of quantum computers. The algorithm implies the ability of such computation based upon the concept of a boolean algebra. Finally we have the ultimate computation for today's quantum computers.

In this section, we first propose herein a novel parallel computation, even though today's algorithm methodology for quantum computing, for all of the combinations of values in variables of a logical function. Our concern so far has been to obtain an attribute of some function. In fact such a task is only for one task problem solving. However, we could treat positively the plural evaluations of some logical function in parallel instead of testing the function for finding out its attribute. In fact, these evaluations of the function are naturally included and evaluated, in parallel, in normal quantum computing discussing a function in a boolean algebra stemmed from atoms in it. As is naturally understandable with mathematics, quantum computing naturally meets the category of a boolean algebra. The reason why we positively introduce a boolean algebra here is because we have multiple evaluations of a function in quantum computing general.

1. Quantum algorithm for determining the 2^2 mappings of a function based on orthogonal states

We propose a quantum algorithm for determining the 2^2 mappings of a function. Suppose newly

$$f : \{0,1\}^2 \to \{0,1\} \tag{25}$$

is a function. We want to know simultaneously the 2^2 mappings $f(0,0), f(0,1), f(1,0)$, and $f(1,1)$. Later we see a complete matching between our results and a boolean algebra F_2. In the boolean algebra F_2, the function is a two-valuable function. For example, $f(x,y)$ is the function where x and y are variables used in mapping f. In what follows, the abbreviation $f(xy)$ stands for $f(x,y)$. We see a combination between a unitary transformation theory and logic theory.

We define the input state as follows, using an application of Deutsch's algorithm expanded:

$$\begin{aligned}|\Psi_0\rangle =&(\frac{1}{\sqrt{2}}|00\rangle|-\rangle_x + \frac{1}{\sqrt{2}}|01\rangle|-\rangle_x)(\frac{1}{\sqrt{2}}|00\rangle|+\rangle_y + \frac{1}{\sqrt{2}}|01\rangle|+\rangle_y)\\ &(\frac{1}{\sqrt{2}}|10\rangle|-\rangle_x + \frac{1}{\sqrt{2}}|11\rangle|-\rangle_x)(\frac{1}{\sqrt{2}}|10\rangle|+\rangle_y + \frac{1}{\sqrt{2}}|11\rangle|+\rangle_y).\end{aligned} \tag{26}$$

From the mapping U_f, we can define the following formulas:

$$U_f|00\rangle|+\rangle_y = \begin{cases}(i)^{f(00)}|00\rangle|+\rangle_y & \text{if} \quad f(00) = 0,\\ (i)^{f(00)}|00\rangle|-\rangle_y & \text{if} \quad f(00) = 1.\end{cases} \tag{27}$$

$$U_f|01\rangle|+\rangle_y = \begin{cases}(i)^{f(01)}|01\rangle|+\rangle_y & \text{if} \quad f(01) = 0,\\ (i)^{f(01)}|01\rangle|-\rangle_y & \text{if} \quad f(01) = 1.\end{cases} \tag{28}$$

$$U_f|10\rangle|+\rangle_y = \begin{cases} (i)^{f(10)}|10\rangle|+\rangle_y & \text{if} \quad f(10) = 0, \\ (i)^{f(10)}|10\rangle|-\rangle_y & \text{if} \quad f(10) = 1. \end{cases} \tag{29}$$

$$U_f|11\rangle|+\rangle_y = \begin{cases} (i)^{f(11)}|11\rangle|+\rangle_y & \text{if} \quad f(11) = 0, \\ (i)^{f(11)}|11\rangle|-\rangle_y & \text{if} \quad f(11) = 1. \end{cases} \tag{30}$$

$$U_f|00\rangle|-\rangle_x = \begin{cases} (-1)^{f(00)}|00\rangle|-\rangle_x & \text{if} \quad f(00) = 0, \\ (-1)^{f(00)}|00\rangle|-\rangle_x & \text{if} \quad f(00) = 1. \end{cases} \tag{31}$$

$$U_f|01\rangle|-\rangle_x = \begin{cases} (-1)^{f(01)}|01\rangle|-\rangle_x & \text{if} \quad f(01) = 0, \\ (-1)^{f(01)}|01\rangle|-\rangle_x & \text{if} \quad f(01) = 1. \end{cases} \tag{32}$$

$$U_f|10\rangle|-\rangle_x = \begin{cases} (-1)^{f(10)}|10\rangle|-\rangle_x & \text{if} \quad f(10) = 0, \\ (-1)^{f(10)}|10\rangle|-\rangle_x & \text{if} \quad f(10) = 1. \end{cases} \tag{33}$$

$$U_f|11\rangle|-\rangle_x = \begin{cases} (-1)^{f(11)}|11\rangle|-\rangle_x & \text{if} \quad f(11) = 0, \\ (-1)^{f(11)}|11\rangle|-\rangle_x & \text{if} \quad f(11) = 1. \end{cases} \tag{34}$$

Applying $U_{f_i}U_{f_i}U_{f_i}U_{f_i}, (i = 0, 1, 2, ..., 2^{2^2} - 1)$ to $|\Psi_0\rangle$, $U_{f_i}U_{f_i}U_{f_i}U_{f_i}|\Psi_0\rangle = |\Psi_1\rangle_i$, therefore leaves us with one of 2^{2^2} cases:

$$\begin{aligned}|\Psi_1\rangle_0 =&(\frac{1}{\sqrt{2}}|00\rangle|-\rangle_x + \frac{1}{\sqrt{2}}|01\rangle|-\rangle_x)(\frac{1}{\sqrt{2}}|00\rangle|+\rangle_y + \frac{1}{\sqrt{2}}|01\rangle|+\rangle_y)\\ &(\frac{1}{\sqrt{2}}|10\rangle|-\rangle_x + \frac{1}{\sqrt{2}}|11\rangle|-\rangle_x)(\frac{1}{\sqrt{2}}|10\rangle|+\rangle_y + \frac{1}{\sqrt{2}}|11\rangle|+\rangle_y)\\ &\text{iff} \quad f_0(00) = 0, f_0(01) = 0, f_0(10) = 0, f_0(11) = 0,\end{aligned} \tag{35}$$

$$\begin{aligned}|\Psi_1\rangle_1 =&(\frac{1}{\sqrt{2}}|00\rangle|-\rangle_x + \frac{1}{\sqrt{2}}|01\rangle|-\rangle_x)(\frac{1}{\sqrt{2}}|00\rangle|+\rangle_y + \frac{1}{\sqrt{2}}|01\rangle|+\rangle_y)\\ &(\frac{1}{\sqrt{2}}|10\rangle|-\rangle_x - \frac{1}{\sqrt{2}}|11\rangle|-\rangle_x)(\frac{1}{\sqrt{2}}|10\rangle|+\rangle_y + i\frac{1}{\sqrt{2}}|11\rangle|-\rangle_y)\\ &\text{iff} \quad f_1(00) = 0, f_1(01) = 0, f_1(10) = 0, f_1(11) = 1,\end{aligned} \tag{36}$$

$$\begin{aligned}|\Psi_1\rangle_2 =&(\frac{1}{\sqrt{2}}|00\rangle|-\rangle_x + \frac{1}{\sqrt{2}}|01\rangle|-\rangle_x)(\frac{1}{\sqrt{2}}|00\rangle|+\rangle_y + \frac{1}{\sqrt{2}}|01\rangle|+\rangle_y)\\ &(-\frac{1}{\sqrt{2}}|10\rangle|-\rangle_x + \frac{1}{\sqrt{2}}|11\rangle|-\rangle_x)(i\frac{1}{\sqrt{2}}|10\rangle|-\rangle_y + \frac{1}{\sqrt{2}}|11\rangle|+\rangle_y)\\ &\text{iff} \quad f_2(00) = 0, f_2(01) = 0, f_2(10) = 1, f_2(11) = 0,\end{aligned} \tag{37}$$

$$\begin{aligned}|\Psi_1\rangle_3 =&(\frac{1}{\sqrt{2}}|00\rangle|-\rangle_x + \frac{1}{\sqrt{2}}|01\rangle|-\rangle_x)(\frac{1}{\sqrt{2}}|00\rangle|+\rangle_y + \frac{1}{\sqrt{2}}|01\rangle|+\rangle_y)\\ &(-\frac{1}{\sqrt{2}}|10\rangle|-\rangle_x - \frac{1}{\sqrt{2}}|11\rangle|-\rangle_x)(i\frac{1}{\sqrt{2}}|10\rangle|-\rangle_y + i\frac{1}{\sqrt{2}}|11\rangle|-\rangle_y)\\ &\text{iff} \quad f_3(00) = 0, f_3(01) = 0, f_3(10) = 1, f_3(11) = 1,\end{aligned} \tag{38}$$

$$\begin{aligned}|\Psi_1\rangle_4 =&(\frac{1}{\sqrt{2}}|00\rangle|-\rangle_x - \frac{1}{\sqrt{2}}|01\rangle|-\rangle_x)(\frac{1}{\sqrt{2}}|00\rangle|+\rangle_y + i\frac{1}{\sqrt{2}}|01\rangle|-\rangle_y)\\ &(\frac{1}{\sqrt{2}}|10\rangle|-\rangle_x + \frac{1}{\sqrt{2}}|11\rangle|-\rangle_x)(\frac{1}{\sqrt{2}}|10\rangle|+\rangle_y + \frac{1}{\sqrt{2}}|11\rangle|+\rangle_y)\\ &\text{iff} \quad f_4(00) = 0, f_4(01) = 1, f_4(10) = 0, f_4(11) = 0,\end{aligned} \tag{39}$$

$$\begin{aligned}|\Psi_1\rangle_5 =&(\frac{1}{\sqrt{2}}|00\rangle|-\rangle_x - \frac{1}{\sqrt{2}}|01\rangle|-\rangle_x)(\frac{1}{\sqrt{2}}|00\rangle|+\rangle_y + i\frac{1}{\sqrt{2}}|01\rangle|-\rangle_y)\\ &(\frac{1}{\sqrt{2}}|10\rangle|-\rangle_x - \frac{1}{\sqrt{2}}|11\rangle|-\rangle_x)(\frac{1}{\sqrt{2}}|10\rangle|+\rangle_y + i\frac{1}{\sqrt{2}}|11\rangle|-\rangle_y)\\ &\text{iff}\quad f_5(00)=0, f_5(01)=1, f_5(10)=0, f_5(11)=1,\end{aligned} \tag{40}$$

$$\begin{aligned}|\Psi_1\rangle_6 =&(\frac{1}{\sqrt{2}}|00\rangle|-\rangle_x - \frac{1}{\sqrt{2}}|01\rangle|-\rangle_x)(\frac{1}{\sqrt{2}}|00\rangle|+\rangle_y + i\frac{1}{\sqrt{2}}|01\rangle|-\rangle_y)\\ &(-\frac{1}{\sqrt{2}}|10\rangle|-\rangle_x + \frac{1}{\sqrt{2}}|11\rangle|-\rangle_x)(i\frac{1}{\sqrt{2}}|10\rangle|-\rangle_y + \frac{1}{\sqrt{2}}|11\rangle|+\rangle_y)\\ &\text{iff}\quad f_6(00)=0, f_6(01)=1, f_6(10)=1, f_6(11)=0,\end{aligned} \tag{41}$$

$$\begin{aligned}|\Psi_1\rangle_7 =&(\frac{1}{\sqrt{2}}|00\rangle|-\rangle_x - \frac{1}{\sqrt{2}}|01\rangle|-\rangle_x)(\frac{1}{\sqrt{2}}|00\rangle|+\rangle_y + i\frac{1}{\sqrt{2}}|01\rangle|-\rangle_y)\\ &(-\frac{1}{\sqrt{2}}|10\rangle|-\rangle_x - \frac{1}{\sqrt{2}}|11\rangle|-\rangle_x)(i\frac{1}{\sqrt{2}}|10\rangle|-\rangle_y + i\frac{1}{\sqrt{2}}|11\rangle|-\rangle_y)\\ &\text{iff}\quad f_7(00)=0, f_7(01)=1, f_7(10)=1, f_7(11)=1,\end{aligned} \tag{42}$$

$$\begin{aligned}|\Psi_1\rangle_8 =&(-\frac{1}{\sqrt{2}}|00\rangle|-\rangle_x + \frac{1}{\sqrt{2}}|01\rangle|-\rangle_x)(i\frac{1}{\sqrt{2}}|00\rangle|-\rangle_y + \frac{1}{\sqrt{2}}|01\rangle|+\rangle_y)\\ &(\frac{1}{\sqrt{2}}|10\rangle|-\rangle_x + \frac{1}{\sqrt{2}}|11\rangle|-\rangle_x)(\frac{1}{\sqrt{2}}|10\rangle|+\rangle_y + \frac{1}{\sqrt{2}}|11\rangle|+\rangle_y)\\ &\text{iff}\quad f_8(00)=1, f_8(01)=0, f_8(10)=0, f_8(11)=0,\end{aligned} \tag{43}$$

$$\begin{aligned}|\Psi_1\rangle_9 =&(-\frac{1}{\sqrt{2}}|00\rangle|-\rangle_x + \frac{1}{\sqrt{2}}|01\rangle|-\rangle_x)(i\frac{1}{\sqrt{2}}|00\rangle|-\rangle_y + \frac{1}{\sqrt{2}}|01\rangle|+\rangle_y)\\ &(\frac{1}{\sqrt{2}}|10\rangle|-\rangle_x - \frac{1}{\sqrt{2}}|11\rangle|-\rangle_x)(\frac{1}{\sqrt{2}}|10\rangle|+\rangle_y + i\frac{1}{\sqrt{2}}|11\rangle|-\rangle_y)\\ &\text{iff}\quad f_9(00)=1, f_9(01)=0, f_9(10)=0, f_9(11)=1,\end{aligned} \tag{44}$$

$$\begin{aligned}|\Psi_1\rangle_{10} =&(-\frac{1}{\sqrt{2}}|00\rangle|-\rangle_x + \frac{1}{\sqrt{2}}|01\rangle|-\rangle_x)(i\frac{1}{\sqrt{2}}|00\rangle|-\rangle_y + \frac{1}{\sqrt{2}}|01\rangle|+\rangle_y)\\ &(-\frac{1}{\sqrt{2}}|10\rangle|-\rangle_x + \frac{1}{\sqrt{2}}|11\rangle|-\rangle_x)(i\frac{1}{\sqrt{2}}|10\rangle|-\rangle_y + \frac{1}{\sqrt{2}}|11\rangle|+\rangle_y)\\ &\text{iff}\quad f_{10}(00)=1, f_{10}(01)=0, f_{10}(10)=1, f_{10}(11)=0,\end{aligned} \tag{45}$$

$$\begin{aligned}|\Psi_1\rangle_{11} =&(-\frac{1}{\sqrt{2}}|00\rangle|-\rangle_x + \frac{1}{\sqrt{2}}|01\rangle|-\rangle_x)(i\frac{1}{\sqrt{2}}|00\rangle|-\rangle_y + \frac{1}{\sqrt{2}}|01\rangle|+\rangle_y)\\ &(-\frac{1}{\sqrt{2}}|10\rangle|-\rangle_x - \frac{1}{\sqrt{2}}|11\rangle|-\rangle_x)(i\frac{1}{\sqrt{2}}|10\rangle|-\rangle_y + i\frac{1}{\sqrt{2}}|11\rangle|-\rangle_y)\\ &\text{iff}\quad f_{11}(00)=1, f_{11}(01)=0, f_{11}(10)=1, f_{11}(11)=1,\end{aligned} \tag{46}$$

$$\begin{aligned}|\Psi_1\rangle_{12} =&(-\frac{1}{\sqrt{2}}|00\rangle|-\rangle_x - \frac{1}{\sqrt{2}}|01\rangle|-\rangle_x)(i\frac{1}{\sqrt{2}}|00\rangle|-\rangle_y + i\frac{1}{\sqrt{2}}|01\rangle|-\rangle_y)\\ &(\frac{1}{\sqrt{2}}|10\rangle|-\rangle_x + \frac{1}{\sqrt{2}}|11\rangle|-\rangle_x)(\frac{1}{\sqrt{2}}|10\rangle|+\rangle_y + \frac{1}{\sqrt{2}}|11\rangle|+\rangle_y)\\ &\text{iff}\quad f_{12}(00)=1, f_{12}(01)=1, f_{12}(10)=0, f_{12}(11)=0,\end{aligned} \tag{47}$$

$$\begin{aligned}|\Psi_1\rangle_{13} =&(-\frac{1}{\sqrt{2}}|00\rangle|-\rangle_x - \frac{1}{\sqrt{2}}|01\rangle|-\rangle_x)(i\frac{1}{\sqrt{2}}|00\rangle|-\rangle_y + i\frac{1}{\sqrt{2}}|01\rangle|-\rangle_y)\\ &(\frac{1}{\sqrt{2}}|10\rangle|-\rangle_x - \frac{1}{\sqrt{2}}|11\rangle|-\rangle_x)(\frac{1}{\sqrt{2}}|10\rangle|+\rangle_y + i\frac{1}{\sqrt{2}}|11\rangle|-\rangle_y)\\ &\text{iff}\quad f_{13}(00)=1, f_{13}(01)=1, f_{13}(10)=0, f_{13}(11)=1,\end{aligned} \tag{48}$$

$$
\begin{aligned}
|\Psi_1\rangle_{14} =& (-\frac{1}{\sqrt{2}}|00\rangle|-\rangle_x - \frac{1}{\sqrt{2}}|01\rangle|-\rangle_x)(i\frac{1}{\sqrt{2}}|00\rangle|-\rangle_y + i\frac{1}{\sqrt{2}}|01\rangle|-\rangle_y) \\
& (-\frac{1}{\sqrt{2}}|10\rangle|-\rangle_x + \frac{1}{\sqrt{2}}|11\rangle|-\rangle_x)(i\frac{1}{\sqrt{2}}|10\rangle|-\rangle_y + \frac{1}{\sqrt{2}}|11\rangle|+\rangle_y) \\
& \text{iff} \quad f_{14}(00) = 1, f_{14}(01) = 1, f_{14}(10) = 1, f_{14}(11) = 0,
\end{aligned} \tag{49}
$$

$$
\begin{aligned}
|\Psi_1\rangle_{15} =& (-\frac{1}{\sqrt{2}}|00\rangle|-\rangle_x - \frac{1}{\sqrt{2}}|01\rangle|-\rangle_x)(i\frac{1}{\sqrt{2}}|00\rangle|-\rangle_y + i\frac{1}{\sqrt{2}}|01\rangle|-\rangle_y) \\
& (-\frac{1}{\sqrt{2}}|10\rangle|-\rangle_x - \frac{1}{\sqrt{2}}|11\rangle|-\rangle_x)(i\frac{1}{\sqrt{2}}|10\rangle|-\rangle_y + i\frac{1}{\sqrt{2}}|11\rangle|-\rangle_y) \\
& \text{iff} \quad f_{15}(00) = 1, f_{15}(01) = 1, f_{15}(10) = 1, f_{15}(11) = 1.
\end{aligned} \tag{50}
$$

Thus, by measuring $|\psi_1\rangle_i$, we may determine simultaneously all the 2^2 mappings of $f_i(x, y)$ for all x and y. This is very interesting indeed: the quantum algorithm gives us the ability to determine a perfect property of $f_i(x, y)$, namely, $f_i(x, y)$ itself. This is faster than a classical apparatus, which would require at least 2^{2^2} evaluations.

Later we discuss the relation between set theory [35] based upon atoms and our results in terms of a boolean algebra. Especially the result reveals a complete matching between quantum computing and a boolean algebra. As is naturally understandable with mathematics, quantum computing belongs to the category of a boolean algebra. We positively mention that the fundamental structures of quantum computing and von Neumann architecture are the same in terms of the category of a boolean algebra. However, the main different is based on parallelism for determining all the mappings used especially in quantum computing.

2. Example using a logical function

Let us consider the case where $i = 1$. The logical function is as follows [35]:

$$f_1(x, y) = A \wedge B. \tag{51}$$

where x and y are variables used in mapping f. $x(= 0, 1)$ is variable for A. $y(= 0, 1)$ is variable for B. We want to evaluate simultaneously all the mappings:

$$f_1(0, 0), f_1(0, 1), f_1(1, 0), f_1(1, 1). \tag{52}$$

In classical case we require 2^{2^2} evaluations. In quantum case we require just one query.

The input state is as folows:

$$
\begin{aligned}
|\Psi_0\rangle =& (\frac{1}{\sqrt{2}}|00\rangle|-\rangle_x + \frac{1}{\sqrt{2}}|01\rangle|-\rangle_x)(\frac{1}{\sqrt{2}}|00\rangle|+\rangle_y + \frac{1}{\sqrt{2}}|01\rangle|+\rangle_y) \\
& (\frac{1}{\sqrt{2}}|10\rangle|-\rangle_x + \frac{1}{\sqrt{2}}|11\rangle|-\rangle_x)(\frac{1}{\sqrt{2}}|10\rangle|+\rangle_y + \frac{1}{\sqrt{2}}|11\rangle|+\rangle_y).
\end{aligned} \tag{53}
$$

Applying $U_{f_1}U_{f_1}U_{f_1}U_{f_1}$ to $|\Psi_0\rangle$, $U_{f_1}U_{f_1}U_{f_1}U_{f_1}|\Psi_0\rangle = |\Psi_1\rangle_1$, we have the following output state:

$$
\begin{aligned}
|\Psi_1\rangle_1 =& (\frac{1}{\sqrt{2}}|00\rangle|-\rangle_x + \frac{1}{\sqrt{2}}|01\rangle|-\rangle_x)(\frac{1}{\sqrt{2}}|00\rangle|+\rangle_y + \frac{1}{\sqrt{2}}|01\rangle|+\rangle_y) \\
& (\frac{1}{\sqrt{2}}|10\rangle|-\rangle_x - \frac{1}{\sqrt{2}}|11\rangle|-\rangle_x)(\frac{1}{\sqrt{2}}|10\rangle|+\rangle_y + i\frac{1}{\sqrt{2}}|11\rangle|-\rangle_y).
\end{aligned} \tag{54}
$$

Therefore we evaluate simultaneously all the mappings of $f_1(x, y)$:

$$f_1(0, 0) = 0, f_1(0, 1) = 0, f_1(1, 0) = 0, f_1(1, 1) = 1. \tag{55}$$

This is faster than a classical apparatus, which would require at least 2^{2^2} evaluations. Likewise, we can evaluate the sixteen functions in a boolean algebra F_2.

3. Relation between the atoms (set theory) and the result in Section III

Let us discuss the relation between the atoms [35] (set theory) and the result in Section III. These A and B are subsets which are constructed using the atoms $f1$ through $f4$ that are disjoint one another. For example, newly using fi as an element of a boolean algebra F_2,

$$\begin{aligned} A &= f1 \vee f3 = \{f1, f3\}, \\ B &= f1 \vee f2 = \{f1, f2\}, \end{aligned} \tag{56}$$

where,

$$\begin{aligned} f1 &= A \wedge B, \\ f2 &= A' \wedge B, \\ f3 &= A \wedge B', \\ f4 &= A' \wedge B'. \end{aligned} \tag{57}$$

We can introduce a boolean algebra F_2 as a power set of the atoms. F_2 is based on the value "1" of the two-variable switching functions. An atom is a function including only one "1" as its mapped value, in the four combinations of the values of A and B for the two-variable function.

Clearly we notice a complete matching between the boolean algebra F_2 and our result in Section III. In fact we can see that Eqs. (36), (37), (39), and (43) are regarded as the four atoms of the boolean algebra F_2. For example, we notice (36) OR operation with (37) is equal to (38) and all elements are derived from the four atoms.

We see that the relation between set theory based upon atoms and our result in terms of a boolean algebra. The important point is that we obtain all the elements of F_2 by means of a power set of atoms when we get the four atoms. Thus we can say that next our aim is of getting simultaneously (36), (37), (39), and (43). That means we get simultaneously (35)–(50) (all sixteen patterns!). This is now possible as we discuss: We can construct very clearly the following quantum state composed on four orthogonal states:

$$|\Psi_1\rangle_1 \otimes |\Psi_1\rangle_2 \otimes |\Psi_1\rangle_4 \otimes |\Psi_1\rangle_8. \tag{58}$$

And we evaluate this quantum state of obtaining all the mappings. Especially, we have a quantum algorithm for evaluating two of logical functions simultaneously [30] and then we have

$$\frac{|\Psi_1\rangle_1 + i|\Psi_1\rangle_2}{\sqrt{2}} \otimes \frac{|\Psi_1\rangle_4 + i|\Psi_1\rangle_8}{\sqrt{2}}. \tag{59}$$

In this case, we evaluate one quantum state of obtaining all the mappings.

We positively stress that the fundamental structures of quantum computing and von Neumann architecture are the same in terms of the category of a boolean algebra. However, the main different is based on parallelism for determining all the mappings used especially in quantum computing. We hope our discussions conclude the very true ultimate importance of the quantum parallelism to construct quantum computers beyond von Neumann architecture.

IV. TOWARD PRACTICALLY MATHEMATICAL EVALUATIONS

From (36), we have

$$\begin{aligned} |\Psi_1\rangle_1 =& (\frac{1}{\sqrt{2}}|00\rangle|-\rangle_x + \frac{1}{\sqrt{2}}|01\rangle|-\rangle_x)(\frac{1}{\sqrt{2}}|00\rangle|+\rangle_y + \frac{1}{\sqrt{2}}|01\rangle|+\rangle_y) \\ & (\frac{1}{\sqrt{2}}|10\rangle|-\rangle_x - \frac{1}{\sqrt{2}}|11\rangle|-\rangle_x)(\frac{1}{\sqrt{2}}|10\rangle|+\rangle_y + i\frac{1}{\sqrt{2}}|11\rangle|-\rangle_y) \\ & \text{iff} \quad f_1(00) = 0, f_1(01) = 0, f_1(10) = 0, f_1(11) = 1. \end{aligned} \tag{60}$$

Hence, we evaluate the mappings of the logical function [35]:

$$f_1(x, y) = A \wedge B. \tag{61}$$

From (41), we have

$$\begin{aligned} |\Psi_1\rangle_6 =& (\frac{1}{\sqrt{2}}|00\rangle|-\rangle_x - \frac{1}{\sqrt{2}}|01\rangle|-\rangle_x)(\frac{1}{\sqrt{2}}|00\rangle|+\rangle_y + i\frac{1}{\sqrt{2}}|01\rangle|-\rangle_y) \\ & (-\frac{1}{\sqrt{2}}|10\rangle|-\rangle_x + \frac{1}{\sqrt{2}}|11\rangle|-\rangle_x)(i\frac{1}{\sqrt{2}}|10\rangle|-\rangle_y + \frac{1}{\sqrt{2}}|11\rangle|+\rangle_y) \\ & \text{iff} \quad f_6(00) = 0, f_6(01) = 1, f_6(10) = 1, f_6(11) = 0. \end{aligned} \tag{62}$$

Hence, we evaluate the mappings of the logical function [35]:

$$f_6(x, y) = \text{Exclusive OR}(A, B). \tag{63}$$

From (42), we have

$$\begin{aligned}|\Psi_1\rangle_7 =&(\frac{1}{\sqrt{2}}|00\rangle|-\rangle_x - \frac{1}{\sqrt{2}}|01\rangle|-\rangle_x)(\frac{1}{\sqrt{2}}|00\rangle|+\rangle_y + i\frac{1}{\sqrt{2}}|01\rangle|-\rangle_y)\\ &(-\frac{1}{\sqrt{2}}|10\rangle|-\rangle_x - \frac{1}{\sqrt{2}}|11\rangle|-\rangle_x)(i\frac{1}{\sqrt{2}}|10\rangle|-\rangle_y + i\frac{1}{\sqrt{2}}|11\rangle|-\rangle_y)\\ &\text{iff}\quad f_7(00) = 0, f_7(01) = 1, f_7(10) = 1, f_7(11) = 1.\end{aligned} \tag{64}$$

Hence, we evaluate the mappings of the logical function [35]:

$$f_7(x, y) = A \vee B. \tag{65}$$

We have studied quantum operations based upon the quantum mechanics. Firstly, we have used Deutsch's algorithm with the usual phase kickback formation to develop the very true overbridging between usual quantum mechanics (and then quantum computing) and a boolean algebra. In this, we have confirmed that usual quantum operations are useful, beyond the quantum computing only for quantum mechanics operations, for very true mathematical evaluations just like an arithmetic operation. We demonstrate two typical arithmetic calculations in the binary system.

As an example of a simple addition $1 + 1$ in the binary system, we are going to develop the process of how to calculate this:

To solve it, fortunately we have a formula here

$$f_6(x, y) = \text{Exclusive OR}(A, B). \tag{66}$$

$$f_1(x, y) = A \wedge B. \tag{67}$$

$$1 + 1 =??. \tag{68}$$

$$\text{Sum} = \text{Exclusive OR}(1, 1) = 0. \tag{69}$$

$$\text{Carry} = 1 \wedge 1 = 1. \tag{70}$$

Hence we have very clearly

$$1 + 1 = 10 \tag{71}$$

according to the algorithm for addition in the binary system. The concrete and specific calculation $(1 + 1)$ is faster than that of a classical apparatus which would require $2^8 = 256$ steps when we introduce only the half adder operation.

In more details, we must use the rule of a half adder that is composed of by using some formulae in the boolean algebra [35]. In the half adder, the function of it is the SUM and a Carry to the next digit position. The circuit consists of two boolean functions (69) and (70).

Further, we could mention a little bit complicated example

$2 + 3$ in the decimal system.

In addition of the half adder operation, we need one more operation the full adder [35]. As for the full adder, it is by the two half adders and the "OR" operation $A \vee B$ $(A, B \in \{0, 1\})$ in the boolean algebra to take out the result from the previous digit. The operation is left here because it is obvious mathematically. Anyhow we can obtain the result 5 in the decimal system.

To solve it, fortunately we have a formula here

$$f_6(x, y) = \text{Exclusive OR}(A, B). \tag{72}$$

$$f_1(x, y) = A \wedge B. \tag{73}$$

$$f_7(x, y) = A \vee B. \tag{74}$$

$$10 + 11 =???. \tag{75}$$

$$\text{Sum} = \text{Exclusive OR}(0, 1) = 1. \tag{76}$$

$$\text{Carry} = 0 \wedge 1 = 0. \tag{77}$$

Thus we have

$$10 + 11 =??1. \tag{78}$$

Also we see

$$\text{Carry } \mathrm{C_i} = 0. \tag{79}$$

Our second aim is of calculating $1+1$ considering Carry $\mathrm{C_i} = 0$ by using a full adder. The first half adder says

$$\text{Exclusive OR}(1,1) = 0. \tag{80}$$
$$\text{Carry} = 1 \wedge 1 = 1. \tag{81}$$

The second half adder says

$$\text{Sum} = \text{Exclusive OR}(\text{Carry } \mathrm{C_i}, \text{Exclusive OR}(1,1)) = 0. \tag{82}$$
$$\text{Carry} = \text{Carry } \mathrm{C_i} \wedge \text{Exclusive OR}(1,1) = 0. \tag{83}$$

Thus we see $10 + 11 =?01$. We have finally the carry Carry $\mathrm{C_0}$ as follows: (This is (81) $\vee$ (83)).

$$\text{Carry } \mathrm{C_0} = 0 \vee 1 = 1. \tag{84}$$

Hence we have very clearly

$$10 + 11 = 101 \tag{85}$$

according to the algorithm for addition in the binary system. The concrete and specific calculation $(2+3)$ is faster than that of a classical apparatus which would require $2^{12} = 4096$ steps when we introduce the full adder operation. The quantum advantage increases when two numbers we treat become very large. Toward practical quantum-gated computers, experimental demonstrations of our argumentations are going to be interested.

V. CONCLUSIONS

In conclusion, we have expanded Deutsch's algorithm for determining all the mappings of a function by using four orthogonal states. Using this, we have proposed a parallel computation for all of the combinations of values in variables of a logical function by using sixteen orthogonal states. As an application of our algorithm, we have demonstrated two typical arithmetic calculations in the binary system. We have studied an efficiency for operating a full adder/half adder by quantum-gated computing. The two typical arithmetic calculations have been $(1+1)$ and $(2+3)$. The typical arithmetic calculation $(2+3)$ has been faster than that of its classical apparatus which would require $2^{16} = 4096$ steps when we introduce the full adder operation. Another typical arithmetic calculation $(1+1)$ has been faster than that of its classical apparatus which would require $2^8 = 256$ steps when we introduce only the half adder operation.

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Ethical Approval

The authors are in an applicable thought to Ethical Approval.

Competing Interests

The authors state that there is no conflict of interest.

Authors' Contributions

Koji Nagata and Tadao Nakamura wrote and read the manuscript.

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No Data associated in the manuscript.

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