ON THE FINITENESS OF SEQUENCES OF EVEN SQUAREFREE FIBONACCI NUMBERS

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Abstract. Let $2p_1p_2\dots p_{k-1}$ be an even squarefree Fibonacci number with k distinct prime factors. For each positive k, such numbers form an integer sequence. We conjecture that each such sequence has only a finite number of terms. In particular, the factorization data for the first 1000 Fibonacci numbers suggests that there is only one such term for $k = 2$, 5 for $k = 3$, and 8 for $k = 4$. We also renew attention to the fact that a proof that there are infinitely many squarefree Fibonacci numbers remains lacking. Some approach to proving this, emerging from our study, is suggested.

1. Introduction

Fibonacci numbers are a well-known sequence of non-negative integers that has been a subject of mathematical study for many centuries now. Previously known to Indian mathematicians, introduced to the West by Fibonacci around 1202, the research on them has accelerated significantly in the last 200 years or so. As a result, they have been found to appear in a number of circumstances both in mathematics and in natural sciences.

The Fibonacci sequence, $f(n)$, that we are concerned with here, is defined by a simple recurrence formula, $f(n) = f(n-1)+f(n-2)$, with $f(1) = f(2) = 1$, which can be naturally extended to allow $f(0) = 0$. The first 12 positive terms of it are 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144. Note that 8 and 144 are powers of 2 and 12, respectively. One can prove that these are the only powers greater than 1 in this sequence, which shows how restrictive it can be in some ways.

One easily notices that the even terms occur twice as less frequently as their odd counterparts. Indeed, they occur for every third value of their index, i.e., $f(3m)$ is always even. It is the even terms of the Fibonacci sequence that are of our main interest here. Specifically, those that are also squarefree, meaning not divisible by a square of any prime number.

While Fibonacci numbers follow a simple pattern, they grow rather fast and that quickly leads to complex issues, one of them being their factorization. It is relatively easy to factorize the first 300 or so Fibonacci numbers with the help of software tools such as PARI/GP [1] or Mathematica, but it gets progressively harder to do so for higher indices of these numbers.

For this, more powerful computing resources are needed. Fortunately, the factorization data for the first 1000 Fibonacci numbers is freely available on Blair Kelly's website [2]. That is the data we used for this work. It appears that the project

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of the factorization of the first 10,000 Fibonacci numbers has yet to be completed; more information about it can also be found at the site mentioned.

This brief paper, written in an expository manner, is organized as follows. In the next section, we present the crux of our work, a certain elementary conjecture that concerns even squarefree Fibonacci numbers. The conjecture is based on the data mentioned above that we analyze in the section that follows subsequently. There, we make additional observations that could also be expressed as conjectures, and whose proof would automatically lead to establishing that there is an infinite number of squarefree Fibonacci numbers. They are not the main objective of this paper, though, and only one of them may be deemed original. Some further discussion is placed in the last section.

2. The conjecture

It seems to be widely believed [3], though still remains unproved, that infinitely many terms of the Fibonacci sequence are squarefree. While this may indeed be the case, the conjecture we present below aims to emphasize that things appear to be somewhat different in this respect for odd and even squarefree Fibonacci numbers.

What we mean by different here can be well illustrated by sphenic Fibonacci numbers, i.e., squarefree Fibonacci numbers that are a product of three distinct primes.

There are 37 such numbers among the first 300 Fibonacci numbers, of which only 5 are even, the last of them corresponding to index number 51. Now, these are not all sphenic numbers among the first 1000 Fibonacci numbers. There are more and the last of them in this range has its index as high as 983. But there are no more even sphenic Fibonacci numbers in this range, none beyond this meager index number of 51.

The situation looks even more drastic for squarefree Fibonacci semiprimes. Among the first 1000 Fibonacci numbers, one finds only one even squarefree semiprime - 34, corresponding to index number 9 - against over 30 odd squarefree semiprimes, the last of which in this range has index number 991.

One should add that odd squarefree semiprime and odd sphenic Fibonacci numbers continue to appear beyond the first thousand.

These and similar circumstances (see the next section) prompted us to put forward a simple conjecture that follows.

Let $2p_1p_2\dots p_{k-1}$ be an even squarefree Fibonacci number with k distinct prime factors. For each positive k, such numbers form an integer sequence.

Conjecture. *Each sequence specified above has only a finite number of terms.*

The conjecture does not imply that the number of all even squarefree Fibonacci numbers be finite. It does not have to be.

The case of $k = 1$ - the sequence in this case consists of 2 only - is trivially correct. We need a proof for $k > 1$.

The conjecture is supported by the data that we will turn to next. What this data suggests, in particular, is that the total number of even squarefree Fibonacci numbers that are a product of 2, 3, and 4 primes is 1, 5, 8, respectively - all single digits.

3. Data Analysis

The data presented below collects the index numbers n and the so called p-index numbers of even squarefree Fibonacci numbers up to $n = 1000$. They are arranged according to k, the number of terms in the even squarefree Fibonacci number.

The p-index number is an auxiliary index number p, such that $n = 3p$, suggesting that p is prime, which appears to be the case very often, for the squarefree numbers of two lowest k's perhaps even exclusively, assuming as the data suggests and as discussed earlier, that their number is limited to single digits.

Indices
$k = 2$: 9.
$k = 3$: 15, 21, 33, 39, 51.
$k = 4$: 27, 87, 93, 123, 249, 321, 393, 453.
$k = 5$: 45, 57, 63, 69, 111, 141, 183, 201, 213, 267, 303, 363, 381, 501, 579, 771.

P-indices
$k = 2$: 3.
$k = 3$: 5, 7, 11, 13, 17.
$k = 4$: 9, 29, 31, 41, 83, 107, 131, 151.
$k = 5$: 15, 19, 21, 23, 37, 47, 61, 67, 71, 89, 101, 121, 127, 167, 193, 257.

As we can see, among those 30 (1+5+8+16) p-index numbers, only 4 (9, 15, 21, 121) are composite.

This suggests that perhaps all Fibonacci numbers whose index number is $3p$, where p is an odd prime, are squarefree (and even). The data up to $n = 1000$, supports this thesis, yet that's only 66 data points. Incidentally, the same kind of observation can be made for Fibonacci numbers with odd prime indices. Here, it too appears that $f(p)$ is exclusively squarefree. There are 167 prime index Fibonacci numbers up to $n = 1000$ and all of them are odd squarefree numbers.

Strangely enough, while these observations are not that hard to make, or so it would seem, they are not necessarily easy to find in the literature of the subject. We managed to locate a mention of the latter on two websites, but not of the former, perhaps because the even squarefree Fibonacci numbers have not been of particular research interest so far.

These observations, while simple, are potentially important for their formal mathematical proof would also prove that there is infinitely many squarefree Fibonacci numbers.

Allow us to note one more thing in passing. To this end, let $N(k)$ be the total number of even squarefree Fibonacci numbers corresponding to k, assuming, as our conjecture claims, that these totals are finite and that the data is exhaustive in these cases - the latter is highly suggestive, but both are unproven. We then see that a certain pattern emerges: $N(k)$ behaves approximately as $2^{(k-1)}$. More data, harder to get, for higher k's, is needed to substantiate this pattern.

4. Conclusion

The conjecture proposed in this paper is both elementary and concerns a widely studied, well-known class of numbers, the Fibonacci sequence. It also concerns squarefree numbers, a fundamental class of numbers, an active field of research as well. Despite all of that, it has apparently not been put forward before, or at least, we have not been able to find a mention of it in the literature of the subject.

It is possible that for some reasons, unknown to us, the conjecture is flat out wrong and hence long dismissed. However, the experimental evidence presented in this paper indicates otherwise.

We have also noted that by proving that all Fibonacci numbers of odd prime indices, p, are squarefree, or that all such numbers with odd indices $3p$, giving rise to even Fibonacci numbers only, are squarefree, one also proves that there is an infinite number of squarefree Fibonacci numbers. Hence, we have two possible general venues, the latter probably lesser known than the former, to prove what is widely believed to be true, but which we have still not managed to establish formally.

It may be worth pointing out that among the first 300 positive Fibonacci numbers, 230 are squarefree, as one can easily find out using PARI/GP. This amounts to about 76.7%. Using tables, one learns that there are 766 squarefree numbers among the first 1000 terms of the Fibonacci sequence with positive indices. This shows that not only do the squarefree terms appear to be dense in the sequence, but that it also happens at a pretty high rate: among natural numbers this rate (understood as asymptotic density) is $\pi^2/6$ or about 60.8%.

Yet, to reiterate the point, a proof that there is an infinite number of squarefree Fibonacci numbers is still lacking, a rather unsettling state of affairs in light of the apparent simplicity of the problem in question. While the main purpose of this paper was to present some conjecture, another was to revive interest in this very issue.

The finite nature of the sequences the conjecture addresses is reminiscent of at least two other phenomena related to the Fibonacci sequence. One, already mentioned earlier, is that there are only two perfect powers greater than 1 (8 and 144) in the entire sequence (see [4]). The other, lesser known, has to do with the refactorable numbers, i.e., numbers divisible by the number of their divisors, such as 1, 2, 8, or 12. It turns out, as shown in [5], that there is only a finite number of refactorable Fibonacci numbers. Exactly six, with index numbers 1, 2, 3, 6, 24, and 48.

It is conceivable that the methods used to prove the finiteness of the number of perfect powers or that of refactorable numbers in the Fibonacci sequence can also be employed for proving or disproving the conjecture presented here.

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References

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