ON DISTRIBUTIONS OF SOME PRIME NUMBERS

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Abstract. This paper focuses on one of the most studied areas of number theory – the distribution of prime numbers. Particularly, the last section of this paper comes up with a conjecture that could lead to a new way to look the distribution of prime numbers.

Introduction and Background

The prime number theorem is one of the fundamental theorems in number theory.

The prime number theorem, for a given value of x, takes π(x) (the prime counting function) to represent the number of primes that are not greater than x.

Theorem 1.1.(Prime number theorem). *The asymptomatic distribution of the prime numbers among the positive integers can be shown by:*

$$\pi(x) = \frac{x}{\log(x)}$$

In this paper, the proof of the prime number theorem is not presented as there are many literatures including many ways of proving it.

In relation to the PNT, one of the most important works on prime numbers can be presented as of a consequential proof i.e. the Bertrand postulate. Bertrand's work opened for lots of other scholarly advancements on the intervals of distributions of prime numbers it the Bertrand postulate (often regarded as 'postulate' but was proved by Tchebychev) is presented as a theorem in this paper.

Theorem 1.2(Bertrand-Tchebychev theorem): *For all $n \geq 1$, there is a prime number, p that is element of* $(n, 2n]$.

N.B.: *This interval in Theorem 1.2 is usually given as* $(n, 2n]$., *that is to include the only even prime number, 2. If the interval of n is given as* $(1, \infty)$, *the interval of Theorem 1.2 follows as* $(n, 2n)$. *Both are equivalent statements.*

In 1845, the French mathematician Joseph Bertrand conjectured Theorem 1.2. After five years, in 1850, Tchebychev proved the conjecture.

This proof of Tchebychev of Bertrand postulate is found on [1]. Elegantly elementary proof of the postulate was provided by Paul Erdos in 1932 is also found on [2].

Bertrand- Tchebychev theorem and its implicative assumptions through Goldbach Conjecture

Goldbach conjecture is one of the most famous open questions in whole of mathematics. In this paper, I am defining 'Goldbach Sum' as follows:

Definition 2.1 (Goldbach Sum): is the sum of two prime numbers.

Conjecture 2.1 (Goldbach Conjecture): Every even natural number, $n > 2$, can be expressed as the sum of two prime numbers.

Now, following from a logical deduction, it is evident that:

Theorem 1.2 implies for extended Bertrand interval, i.e., the theorem is also consistently true for $n \to 2n$ and $2n \to 4n$.

Theorem 2.1: (*Suppose Goldbach conjecture to be true.) Then, the set of all prime numbers that are between two unique Goldbach sums are infinitely many.* (Or, *between any two unique Goldbach sums, there is at least one prime number).*

Proof 2.1: Elementary Proof of Theorem 2.1.

From Theorem 1.2: For $n>1$, $n<p<2n$ implies for $n>1$, $2n<p<4n$. [1.2.1.]

Theorem 2.1 assumes Goldbach conjecture to be true. Therefore, the following holds:

$2n= r+s$ and $4n= a +b$, where r, s, a, and b are unique odd prime numbers. [1.2.2]

From [1.2.1.] and [1.2.2.], the following is deduced:

$\mathbf{r+s < p < a+b.}$ ■

By proof 2.1, I have successfully shown that any possible proof of Goldbach conjecture makes Theorem 2.1 a corollary.

The intervals regarding to the squares of prime numbers

Lemma 3.1: *For any prime number not less than 5, $p \geq 5$, p^2-1 is always a multiple of 24.*

Proof 3.1: Common proof of Lemma 3.1

$P^2-1 = (p+1)(p-1)$ and we know that every prime number with the specified interval is an odd number. Therefore: $P^2-1 = (2n+1)^2 - 1$.

- $(2n+1)^2 - 1 = 4n^2+4n$
- $4n^2+4n = 4n(n+1) \ldots\ldots\ldots (n \in 2k) \vee ((n+1)) \in 2k) \equiv TT \ldots k \in \{1, 2, 3 \ldots\}$ Implies that: $P^2 - 1 = 8k \ldots k \in \{1, 2, 3 \ldots\}$ ……………….. [3.1.1]

P, P +1, P − 1 are consecutive numbers. And P is a prime number. Therefore, the following holds true:$((P+1) \in 3k) \vee ((P-1)) \in 3k) \equiv T$ . . . k ∈ {1, 2, 3 . . .} .... [3.1.2.]

From [3.1.1.] and [3.1.2.], it is evident that, $P^2 - 1 = 24(k)$ . . .. k ∈ {1, 2, 3 . . .} ■

By proof 3.1, Lemma 3.1 is proved. And from Lemma 3.1, I have deduced the following theorem which I proved in proof 3.2.

Theorem 3.1.: *There are infinite sets of triads of odd prime numbers, p, q and r, which fulfill the following condition:*

$$P^2 - q^2 = r^2 - 1 \text{............... where } P > q$$

Proof 3.2: Elementary proof of Theorem 3.1

From Lemma 3.1, it directly follows that, for two prime numbers p and q, $p^2 - 1$ and $q^2 - 1$ are multiples of 24.

Let $p^2 - 1 = 24(x)$ and $q^2 - 1 = 24(y)$

Then, $p^2 - q^2 = 24(x - y)$, $p^2 - q^2 = 24w$ . . . where w= x-y.. [3.2.1.]

Let set A be the set of all multiples of 24 i.e. {24, 48, 72........24n}

From all elements of set A, those elements that give prime numbers upon the addition of one are called *vicinus primus* (neighbors of primes).

Let set B be the set of all *vicinus primus.* As it is obvious, set B is subset to set A.

Let set C be the set of differences of different elements of set B, i.e., all possible values of 'w'. Elements of set C that are in set B are, therefore, *vicinus primus.* This can be restated as the following:

$ww \in (B \cap C)$ Implies $24(w) + 1 = r^2$ . . . where **r** is a prime number . . . . . . [3.2.2.] ■

From [3.2.1.] and [3.2.2.], it is shown that $P^2 - q^2 = r^2 - 1$ where $P > q$.

Summative and Subtractive Intervals of Prime Numbers

New technical terms I have used on section 4 are defined as follows.

N.B.: Goldbach's Sum (As defined in Definition 2.1): The sum of two prime numbers.

Definition 4.1 (Goldbach's Sequence): is a sequence of Goldbach's sums.

- Partially listed elements of Goldbach's sequence: 5, 8, 12, 18, 24 . . .

More clearly, Goldbach's sequence is equivalent to the following expression:

$$\mathbf{G_N = (P_N + P_{N+1}), (P_{N+1} + P_{N+2}), (P_{N+2} + P_{N+3}) \ldots (P_{N+K} + P_{N+(K+1)})}$$

Where N and K are any natural number

P_N is the N^{th} prime number

Definition 4.2 (De Polignac's Difference): is the difference of two prime numbers.

Definition 4.3 (De Polignac's Sequence): is a sequence of Polignac's difference.

- Partially listed elements of Polignac's sequence: 1, 2, 2, 4, 2, 4, 2, 4, 6 . . .

More clearly, Goldbach's sequence is equivalent to the following expression:

$$\mathbf{D_N = (P_{N+1} - P_N), (P_{N+2} - P_{N+1}), (P_{N+3} - P_{N+2}) \ldots (P_{N+(K+1)} - P_{N+K})}$$

Where N and K are any natural number

P_N is the N^{th} prime number

After my work on intervals of prime numbers, I have put forward the following conjecture:

Conjecture 4.0 (Ali's Conjecture): *Between a Goldbach sum and De Polignac's difference of the same prime numbers, p and q, there exist a prime number, r, that is different from both p and q.*

To rephrase my conjecture: From any two prime numbers, p and q, there exists a prime number, r, which satisfies the following expression:

$$r \in (D(p, q)\, , G(p, q)) \text{ or}$$

$$r \in (p - q, p + q) \text{ and } r \neq p, q$$

Lazarus' First Conjecture

A New Development on Number Theory and the Twin Prime Conjecture

Introduction

The Twin Prime Conjecture is one of the oldest conjectures in not only number theory but also in whole of mathematical sciences. This conjecture famously states that there are infinite pairs of prime numbers whose difference is two. So far, major breakthroughs have been made by various mathematicians around the world, but whole edging solution has stayed out of sight.

In this section of the paper, I will present a new conjecture that makes the twin prime conjecture merely a consequence of its potential proof. If this conjecture, namely the 'Lazarus' first conjecture, is proven, the twin prime conjecture will automatically become a corollary that is directly deducible from my conjecture.

Lazarus' First Conjecture: states that there exist infinite sets of four prime numbers p, q, r, and s, that are related in the following way:

q is the p'th odd number, r is the q'th odd number, and that $r - q - p = s$.

The orderly sequence of prime numbers that can fill the place of S under a given interval are called "Rachel's Sequence" and the prime numbers that make up this sequence are called Rachel's primes.

For example, 17 is a Rachel prime. Because one can find three prime numbers that are related to it in the aforementioned numeric relation.

These prime numbers are 19, 37, and 73. 37 is the 19^{th} odd number and 73 is the 37^{th} odd number. And $73 - (19 + 37) = 17$.

<u>Computer Language Assisted Analysis of Lazarus' Conjecture</u>

The following python program was used to analyze how far this conjecture:

```
T=[*Comma Separated Values of all Prime numbers less than N*]………N = 1,000,000
S=[  ]
for i in T:
    x=(2*i)-1
    if x in T:
        y=(2*x)-1
        if y in T:
            S.append(y)
#print(S)
h=[]
f=[]


for d in S:
    j=(d+1)/2
    l=(j+1)/2
    a=d-j-l
    if a in T:
        f.append(a)

print(f)
print(len(f))
#print(T)
```

And, as an outcome of this code, I have found about 118 Rachel primes for all primes < 1,000,000. All 118 Rachel's prime numbers for P <1,000,000 are given below as comma listed values.

17.0, 617.0, 827.0, 1277.0, 2087.0, 2129.0, 2309.0, 2789.0, 3767.0, 4157.0, 4229.0, 4259.0, 4637.0, 5417.0, 7559.0, 8627.0, 13679.0, 15287.0, 16649.0, 17027.0, 17837.0, 18119.0, 19139.0, 20639.0, 20807.0, 21587.0, 25409.0, 26699.0, 28547.0, 29207.0, 29669.0, 31769.0, 32117.0, 32189.0, 32717.0, 33179.0, 34847.0, 44087.0, 44129.0, 46817.0, 47657.0, 48779.0, 49787.0, 49937.0, 51197.0, 53147.0, 55217.0, 55817.0, 57329.0, 57557.0, 58169.0, 61379.0, 61559.0, 63647.0, 63839.0, 64919.0, 67427.0, 70139.0, 70619.0, 71807.0, 78539.0, 80489.0, 81929.0, 82349.0, 87629.0, 92219.0, 97847.0, 99707.0, 104849.0, 108107.0, 108377.0, 109199.0, 109469.0, 112337.0, 115979.0, 117809.0, 119099.0, 121577.0, 125219.0, 130649.0, 131249.0, 131477.0, 131837.0, 132707.0, 151607.0, 152039.0, 156059.0, 156257.0, 159167.0, 164447.0, 164837.0, 168449.0, 170099.0, 170759.0, 174989.0, 175757.0, 179819.0, 187067.0, 189797.0, 193859.0, 195047.0, 195929.0, 199739.0, 202637.0, 203657.0, 211049.0, 223919.0, 224069.0, 226199.0, 228509.0, 229589.0, 231269.0, 233939.0, 239429.0, 240257.0, 247337.0, 248639.0, 249857.0.

Relationship between Lazarus' conjecture and the Twin prime conjecture

In this section, I am going to present an elementary proof that proves that the twin prime conjecture is can be proved by default if Lazarus' conjecture can be proved, i.e., for a possible Lazarus' theorem, the twin prime conjecture serves as a corollary.

Proposition 5.0 (On the Generalization of solutions for declarations of infinite size): If set A is a sub set of set B and if set A is proved to have a size of X, then, it is necessary that set B has at least the size of X.

$$\text{i.e., } A \subseteq B \Rightarrow n(B) \geq n(A)$$

If one is able to show that a pair of the four prime numbers of Lazarus' conjecture are subsets of set of twin primes, and if we assume the suggested conjecture to be true, it would be necessary to conclude that the twin prime conjecture is a direct consequence for the Lazarus' conjecture.

Theorem 5.0 (On the relationship between the twin prime and the Lazarus' conjecture): A mathematically deducted solution of Lazarus' conjecture implies a solution for the twin prime conjecture.

Elementary proof for Theorem 5.0:

Let the four prime numbers in Lazarus' conjecture be p, q, r and s.

i.e., $q = 2p - 1$ 1

$r = 2q - 1$ 2

$s = r - (p + q)$ 3

Substituting the assigned relationships of equation 1 and 2 inside equation 3, we get:

$s = 2q - 1 - (p + 2p - 1)$

$= 2(2p - 1) - 1 - (3p - 1)$

$= 4p - 2 - 1 - 3p + 1$

$s = p - 2$ ■

By this, we have shown that two of the four prime numbers involved in the Lazarus' conjecture are twin prime numbers. Hence, supposing Lazarus' conjecture to be true, it appears an absolute mathematical necessity to conclude that twin primes are infinite.

Additional Contributions of the Lazarus' conjecture.

A possible proof of the Lazarus' conjecture can also be used for proving the infinite number of Sophie-Germain primes at primes that create overlapping cases of Lazarus conjecture with the twin prime conjecture. If Rachel's primes are infinite in size, so are couple of prime numbers, p and r, where $r = 2p - 1$. Hence, intersecting points of the two conjectures appear at a point where another prime number w, that can be substituted to the value of $2p + 1$, exists. So, at points where w exists, both twin prime numbers and Sophie Germain prime numbers appear.

Considering that Rachel's primes are infinitely existing, so are couple of prime numbers, p and r, where $r = 2p - 1$. Hence, from the infinitely standing twin prime numbers, those that create a twin prime relation with the prime number, r, are either in the form of $2p - 3$ or $2p + 1$. The latter is what we call a format of Sophie – Germain prime numbers. In this regard, the Lazarus' conjecture creates link between two very famous problems: the twin prime and the Sophie-Germain problem.

Moreover, from equation 1 of theorem 1.0, by substituting the final value of s, which is $p - 2$, unto p, we get the prime number $q = 2s - 3$. Again, this shows that, supposing the Lazarus' conjecture true, it is mathematically evident that there are infinite number of three prime numbers, p, r, and q, that $2q - 1 = p$ and $2q + 1 = r$.

The role of Rachel's primes are also profound in the world of data encryption.

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