Conjecture that no sum of divisors exists of the form $x^2 + 1$

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Abstract

In this paper, we conjecture that no sum of divisors exists of the form $x^2 + 1$. We offer a proof only for prime numbers, showing that prime numbers cannot have a sum of divisors that is of the form $x^2 + 1$.

0.1 Introduction

The divisor function, $\sigma_k(n)$, for any integer, n, is defined as the sum of the kth powers of the integer divisors of n, i.e. d, and represented as:

$$\sigma_k(n) = \sum_{(d|n)} d^k.$$

When $k = 1$ the divisor function is called the sigma function, sometimes denoted as $\sigma_1(n)$ but conventionally and more often denoted simply as $\sigma(n)$. Here we only consider the case $k = 1$.

The first few values of $\sigma(n)$, where $\sigma(1) = 1$, are (OEIS: A000203):
$1, 3, 4, 7, 6, 12, 8, 15, 13, 18, 12, 28, 14, 24, 24, 31, 18, 39, 20, 42, 32, 36, 24, 60, 31, 42, 40...$

Where r is the number of distinct prime factors of n, p_i is the ith prime factor, and a_i is the maximum power of p_i by which n is divisible:

$$\sigma(n) = \prod_{i=1}^{r} \frac{p_i^{(a_i+1)} - 1}{p_i - 1}. \tag{1}$$

The divisor function is multiplicative (since each divisor c of the product mn with $\gcd(m, n) = 1$ distinctively correspond to a divisor a of m and a divisor b of n), but not completely multiplicative:

$$\gcd(a, b) = 1 \Longrightarrow \sigma(ab) = \sigma(a)\sigma(b). \tag{2}$$

When n is prime p, then

$$\sigma(p) = p + 1. \tag{3}$$

Finally, it is known that the sum of divisors is only ever odd if n is a square, x^2, or twice a square, $2x^2$.

However, a study of the values of n up to 30 shows that there are no sums of divisors of the form $x^2 + 1$.

Alternatively stated, in the following OEIS sequence, which describes $\sigma(n) - 1$, $a(0) = 0$, $a(1) = 0$, there are no values equal to a square:
$0, 0, 2, 3, 6, 5, 11, 7, 14, 12, 17, 11, 27, 13, 23, 23, 30, 17, 38, 19, 41, 31, 35, 23, 59, 30, 41, 39, 55, 29, 71, 31, 62, 47, 53, 47, 90, 37, 59, 55, 89, 41, 95, 43, 83, 77, 71, 47, 123, 56, 92, 71, 97, 53, 119, 71, 119, 79, 89, 59, 167, 61, 95, 103, 126, 83, 143, 67, 125...$

Using this sequence, it is straightforward enough to see why this is true for prime n, but not for composite n.

From this sequence, we can see that

$$\sigma(p^2) = \sigma(p)^2 + \sigma(p) \tag{4}$$

$$\Rightarrow \sigma(p^2) = \sigma(p)[\sigma(p) + 1] \tag{5}$$

The right hand side can never be a square.

For composite n, we can see that:

$$\sigma(ab) = \sigma(a)\sigma(b) + \sigma(a) + \sigma(b). \tag{6}$$

But showing that this cannot be a square is not so easy.