Euler's constant

by

Edgar Valdebenito

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Abstract

The constant $\gamma = 0.5772157\ldots$ is the limit as n tends to infinity of the difference between the sum of n terms of the harmonic series and the natural logarithm of n . It is not know whether this number is algebraic or transcendental, nor even whether it is rational or irrational.

Introduction

The Euler-Mascheroni constant is defined by

$$\gamma = \lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}-\ln(n)\right)=\lim_{n\to\infty}\left(H_n-\ln(n)\right)=0.5772157\ldots$$

In this note we give some formulas related to γ.

Formulas

Entry 1.

$$z_0=0.07\ ,\ z_{n+1}=2\tanh^{-1}\left(1+\frac{z_n}{\ln(z_n)}\int_1^\infty\frac{\ln(x)}{1+\cosh(z_n x)}\,\mathrm{d}x\right)\ ,\ n=0,1,2,3,\ldots$$

$$\lim_{n\to\infty} z_n = z = 0.068202792347088794606152925831653 8\ldots$$

Entry 2.

$$\gamma=\ln\left(\frac{\pi}{2}\right)-\ln(z)\tanh\left(\frac{z}{2}\right)+\sum_{n=1}^{\infty}\frac{(-1)^{n-1}2\left(2^{2n}-1\right)B_n\, z^{2n-1}}{(2n)!\,(2n-1)}$$

where

$$B_n=\left\{\frac{1}{6},\frac{1}{30},\frac{1}{42},\frac{1}{30},\frac{5}{66},\frac{691}{2730},\ldots\right\}$$

remark: B_n are the Bernoulli numbers.

Entry 3.

$$\gamma = \ln\left(\frac{\pi}{2}\right) - \ln(z)\tanh\left(\frac{z}{2}\right) + \int_0^{\frac{z}{2}} \frac{\tanh x}{x}\, dx$$

Entry 4. for t = $\dfrac{-\ln(z)}{1+\cosh z} = 1.341074\ldots$, we have

$$\gamma = \ln\left(\frac{\pi}{2}\right) - \frac{z\ln(z)}{1+\cosh z} + \int_t^{\infty} e^{-x - x\cosh e^{-x - x\cosh e^{-x - \ldots}}}\, dx$$

Entry 5.

$$\int_1^{\infty} \frac{\ln(x)}{1+\cosh(z x)}\, dx = \frac{1}{2}\int_1^{\infty} \ln(x)\left(\operatorname{sech}\left(\frac{z x}{2}\right)\right)^2 dx$$

$$\int_1^{\infty} \frac{\ln(x)}{1+\cosh(z x)}\, dx = \frac{\ln(2)}{z}\left(1 - \tanh\left(\frac{z}{2}\right)\right) + \int_{\frac{1}{2}}^{\infty} \ln(x)\,(\operatorname{sech}(z x))^2\, dx$$

Entry 6.

$$\gamma = \ln\left(\frac{\pi}{2}\right) + \sum_{n=0}^{\infty} (n+1)\, 2^{-n-1} u_n$$

where

$$u_n = \sum_{m=0}^{\infty} \frac{(-1)^m z^{m+1}}{(m+1)!}\left(\frac{1}{m+1} - \ln(z)\right)\sum_{k=0}^{n} (-1)^k \binom{n}{k} (k+1)^m$$

Entry 7.

$$\frac{1}{2}\left(\ln\left(\frac{\pi}{2}\right) - \gamma\right) = \int_0^1 \frac{1}{(1+x)^2}\ln\left(\ln\left(\frac{1}{x}\right)\right) dx$$

Entry 8.

$$\frac{1}{2}\left(\ln\left(\frac{\pi}{2}\right) - \gamma\right) = \int_1^{\infty} \frac{1}{(1+x)^2}\ln(\ln(x))\, dx$$

Entry 9.

$$\frac{1}{2}\left(\ln\left(\frac{\pi}{2}\right) - \gamma\right) = \int_0^{\infty} \frac{e^x \ln(x)}{(1+e^x)^2}\, dx$$

Entry 10.

$$\frac{1}{2}\left(\ln\left(\frac{\pi}{2}\right) - \gamma\right) = \int_{-\infty}^{\infty} \frac{x\, e^{x+e^x}}{\left(1+e^{e^x}\right)^2}\, dx = -\int_{-\infty}^{\infty} \frac{x\, e^{-x+e^{-x}}}{\left(1+e^{e^{-x}}\right)^2}\, dx$$

Entry 11.

$$\left(\ln\left(\frac{\pi}{2}\right)-\gamma\right)=\frac{1}{2}\int_0^\infty \frac{\ln(x)}{\left(\cosh\left(\frac{x}{2}\right)\right)^2}\,dx=\int_0^\infty \frac{\ln(2x)}{(\cosh x)^2}\,dx$$

Entry 12.

$$\frac{1}{2}\left(\ln\left(\frac{\pi}{2}\right)-\gamma\right)=\int_0^\infty \frac{\sinh x}{(\cosh x)^3}\ln(\ln(\cosh(2x)))\,dx$$

Entry 13.

$$\frac{1}{2}\left(\ln\left(\frac{\pi}{2}\right)-\gamma\right)=\int_{\frac{1}{2}}^1 \ln\left(\ln\left(\frac{x}{1-x}\right)\right)dx$$

Entry 14.

$$\frac{1}{2}\left(\ln\left(\frac{\pi}{2}\right)-\gamma\right)=\int_0^{\frac{1}{2}} \ln\left(\ln\left(\frac{1}{x}-1\right)\right)dx$$

Entry 15.

$$\frac{1}{2}\left(\ln\left(\frac{\pi}{2}\right)-\gamma\right)=\int_0^\infty \left(\frac{3}{2}-\frac{1}{1+e^{-e^x}}-\frac{1}{1+e^{-e^{-x}}}\right)dx$$

Entry 16.

$$\frac{1}{2}\left(\ln\left(\frac{\pi}{2}\right)-\gamma\right)=\int_0^\infty \frac{1}{1+e^{e^x}}\,dx-\int_0^\infty \left(\frac{1}{1+e^{-e^{-x}}}-\frac{1}{2}\right)dx$$

Entry 17.

$$\frac{1}{2}\left(\ln\left(\frac{\pi}{2}\right)-\gamma\right)=\int_0^\infty \frac{1}{1+e^{e^x}}\,dx-\frac{1}{2}\int_0^\infty \tanh\left(\frac{e^{-x}}{2}\right)dx$$

Entry 18.

$$\frac{1}{2}\left(\ln\left(\frac{\pi}{2}\right)-\gamma\right)=\int_0^1 \frac{1}{(1+e^{1/x})\,x}\,dx-\frac{1}{2}\int_0^1 \frac{1}{x}\tanh\left(\frac{x}{2}\right)dx$$

Entry 19.

$$\frac{1}{2}\left(\ln\left(\frac{\pi}{2}\right)-\gamma\right)=\int_0^1 \frac{1}{(1+e^{1/x})\,x}\,dx-\frac{1}{2}\int_0^{1/2} \frac{\tanh x}{x}\,dx$$

Entry 20.

$$\frac{1}{2}\left(\ln\left(\frac{\pi}{2}\right)-\gamma\right)=\int_1^\infty \frac{1}{(1+e^{x})\,x}\,dx-\frac{1}{2}\int_0^{1/2} \frac{\tanh x}{x}\,dx$$

Entry 21.

$$\frac{1}{2}\left(\ln\left(\frac{\pi}{2}\right)-\gamma\right)=\sum_{n=0}^\infty (-1)^n\,\Gamma(0,n+1)-\sum_{n=1}^\infty \frac{(-1)^{n-1}(2^{2n}-1)B_n}{(2n-1)\cdot(2n)!}$$

remark: $\Gamma(0, n+1) = \int_1^\infty \frac{e^{-(n+1)x}}{x}\, dx$, is the incomplete Gamma function.

remark: B_n is the Bernoulli numbers.

Entry 22.

$$\frac{1}{2}\left(\ln\left(\frac{\pi}{2}\right) - \gamma\right) = \sum_{n=0}^{\infty}(-1)^n \Gamma(0, n+1) - \sum_{n=0}^{\infty}\frac{2}{(2n+1)\pi}\tan^{-1}\left(\frac{1}{(2n+1)\pi}\right)$$

Entry 23.

$$\frac{1}{2}\left(\ln\left(\frac{\pi}{2}\right) - \gamma\right) = \int_1^\infty \frac{1}{(1+e^x)\,x}\, dx - \frac{1}{2}\int_1^e \frac{x-1}{x\,(x+1)\ln(x)}\, dx$$

Entry 24.

$$z_0 = 0.07\ , \quad z_{n+1} = z_n - \frac{50}{67}\int_{z_n}^{\infty}\frac{\ln(x)}{1+\cosh x}\, dx\ , \ n = 0, 1, 2, 3, ...$$

$$\lim_{n\to\infty} z_n = z = 0.0682027\ ...$$

Entry 25.

$$z_0 = 0.07\ , \ z_{n+1} = 2\tanh^{-1}\left(1 + \frac{1}{\ln(z_n)}\int_0^\infty \frac{\ln\left(1+\frac{x}{z_n}\right)}{1+\cosh(x+z_n)}\, dx\right)\ , n = 0, 1, 2, 3, ...$$

$$\lim_{n\to\infty} z_n = z = 0.0682027\ ...$$

Entry 26.

$$\gamma = \ln\left(\frac{\pi}{2}\right) - \ln(z)\tanh\left(\frac{z}{2}\right) + \frac{2}{z}\ln\left(\cosh\left(\frac{z}{2}\right)\right) + \int_0^{\frac{z}{2}}\frac{\ln(\cosh x)}{x^2}\, dx$$

$$\gamma = \ln\left(\frac{\pi}{2}\right) - \ln(2)\tanh\left(\frac{z}{2}\right) - \int_0^{\frac{z}{2}}\ln(x)(\operatorname{sech} x)^2\, dx$$

Entry 27.

$$v_{n,m} = \sum_{k=0}^{n}(-1)^k\binom{n}{k}(k+1)^m \quad , n, m \in \{0, 1, 2, 3, ...\}$$

$$v_{2n,m} \geq 0 \ \ , n \in \{0, 1, 2, 3, ...\}$$

$$v_{2n+1,m} \leq 0 \ \ , n \in \{0, 1, 2, 3, ...\}$$

$$v_{0,m} = 1 \ \ , m \in \{0, 1, 2, 3, ...\}$$

$$v_{n,m} = 0 \ \ , n \in \{1, 2, 3, 4, ..\}\ , m = 0, 1, 2, ..., n-1$$

Entry 28. $v_{n, m}$, $n, m \in \{0, 1, 2, 3,...\}$

n ; m	0	1	2	3	4	5	6
0	1	1	1	1	1	1	1
1	0	−1	−3	−7	−15	−31	−63
2	0	0	2	12	50	180	602
3	0	0	0	−6	−60	−390	−2100
4	0	0	0	0	24	360	3360
5	0	0	0	0	0	−120	−2520
6	0	0	0	0	0	0	720

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