ON THE IRRATIONALITY OF ALL RIEMANN ZETA FUNCTIONAL VALUES AT ODD INTEGERS A NEW YET SIMPLE APPROACH

Shubhayan Ghosal
Department of Mathematics
Jadavpur University
Email: shubhayanghosal17@gmail.com

Abstract. In this article we provide a proof of the irrationality of $\zeta(2n+1)$ $\forall n \in \mathbb{N}$.Also,in our attempt, we construct an upper bound to the Zeta values at odd integers.It is interesting to see how the irrationality of Zeta values at even positive integers mixed up with Dirichlets irrationality criterion and this bound accelerates our proof further,case by case.

1. Introduction

Until now, there has been proof of irrationality of the Aperys constant only,proved by Roger Apery [1]in 1978 and an elementry proof using double and triple integrals, Legendre polynomial and an algebraic inequality to satisfy the Dirichlet criterion for irrational numbers that came into limelight through F.Beukers [2]. After that many attempts were made to use Beukers type integrals to obtain a similar statement for $\zeta(5)$ but the result was only failure. Thomas Sauvaget [3] even tried to extend his ideas to a multidimensional integrals of unit hypercubes - an extension to Beukers process but he couldn't obtain any plausible upperbound. In 2016, the breakthrough came by the same author that he proved all $\zeta(n)$ are irrational by the same method using polygorathmic approaches. It is noteworthy that great advances in this area are the Rivoals work [4] and Zudilins work [5] that infinitely many $\zeta(2n+1)$ are irrational and that atleast one of $\zeta(5), \zeta(7), \zeta(9), \zeta(11)$ is an irrational number. In view of all these achievements, we devise a completely new, yet simple method to show that all $\zeta(2n+1)$ are irrational numbers.We try to develop the irrationality of $\zeta(2n+1)$ fr the irrationality of $\zeta(2n)$ which is very much known. We just invoke a simpler triangular inequality at the end to end our proof, the rest is just the flow of simple logics derived from known results.

2. The Proof

2.1. The Few Results we would require. We hete present 3 Important Results without proof(as these are already well established) that we shall require in our proof. The Riemann Zeta Function is defined as

$$\zeta(s) = \sum_{k=0}^{\infty} k^{-s} \tag{1}$$

Euler's relation for Riemann Zeta function at even positive integers is

$$\zeta(2n) = \frac{(2\pi)^{2n}}{2(2n)!} B_{2n} \tag{2}$$

Dirichlet criterion [7] for irrational numbers For $\alpha \in \mathbb{R}/\mathbb{Q} \iff \forall \epsilon > 0 \; \exists$ infinitely many $p \in \mathbb{N}$ and $q \in \mathbb{Z}^*$ such that $|\alpha - \frac{p}{q}| < \epsilon$.Here $\mathbb{Z}^*$ means Set of integers excluding 0.

2.2. The Theorem.

Theorem 2.1. *All $\zeta(2n+1)$ are irrational numbers.*

Proof. Consider the following

$$\zeta(2n+1) = 1 + \frac{1}{2^{2n+1}} + \frac{1}{3^{2n+1}} + ...$$

Manipulating it to induce the even powers, we have

$$\zeta(2n+1) = (1 + \frac{1}{2^{2n}} + \frac{1}{3^{2n}} + ...) - (1 + \frac{1}{2^{2n+1}} + \frac{2}{3^{2n+1}} + ...) + 1$$

Now, $\zeta(2n+1) < 1 + \frac{1}{2^{2n+1}} + \frac{2}{3^{2n+1}} + ...$ which implies

$$\zeta(2n+1) < \zeta(2n) + 1 - \zeta(2n+1)$$

or better as

$$\zeta(2n+1) < \frac{1 + \zeta(2n)}{2} = \xi(n) \tag{3}$$

Using (2), we have

$$\zeta(2n+1) < \frac{1}{2} + \frac{(2\pi)^{2n}}{4(2n)!} B_{2n}$$

And as B_{2n} is a rational number, we have

$$\zeta(2n+1) < \frac{1}{2} + \pi^{2n}\psi(n)$$

Where $\psi : \mathbb{N} \longrightarrow \mathbb{Q}, \psi(n) = \frac{2^{2n}B_{2n}}{4(2n)!}$. As there is no polynomial p such that $p(e) = 0$ for $p \in \mathbb{Z}[x]$, owing to the fact that e is transcendental(A direct Corollary of Lindemann-Weierstrass's Theorem),we can confirm that e^{α} is transcendental, where α is an algebraic number.Supposing π to be algebraic and hence πi, it seems to contradict the Euler Identity $e^{\pi i} = -1$,which is not transcendental. Hence,π is a transcendental number.
In short,

$$\zeta(2n) = \pi^{2n}\psi(n)$$

is irrational. It similarly follows that $\xi(n)$ is also an irrational number. Hence from Dirichlet's criterion we can ascertain that $\forall \epsilon > 0\ \exists$ infinitely many $p \in \mathbb{N}$ and $\exists q \in \mathbb{Z}^*$ such that $|\xi(n) - \frac{p}{q}| < \epsilon$. An equivalent statement which can also be proven to imply this is that **As $\xi(n)$ is an irrational number then $\forall \epsilon > 0\ \exists$ infinitely many $r \in \mathbb{Q}$ such that**

$$|\xi(n) - r| < \epsilon \tag{4}$$

Now, setting $\zeta(2n+1) + t_n = \xi(n)$, We move on to 2 distinct cases since we do not know the arithmetic nature of $\zeta(2n+1)$.

2.3. t_n **is rational.** In this case, using (4),we have $\forall\ \epsilon > 0\ \exists r_n \in \mathbb{Q}$ such that

$$|\zeta(2n+1) - r_n| < \epsilon$$

, where

$$r_n = r - t_n$$

,thus rendering $\zeta(2n+1)$ as irrational.

2.4. t_n **is irrational.** In this case we have 2 sets of inequalities, as $\forall\ \epsilon' > 0$ $\exists$ infinitely many $r' \in \mathbb{Q}$ such that

$$|t_n - r'| < \epsilon' \Longrightarrow |-t_n + r'| < \epsilon'$$

$\forall\ \epsilon > 0\ \exists$ infinitely many $r \in \mathbb{Q}$ such that

$$|t_n + \zeta(2n+1) - r| < \epsilon \tag{5}$$

And hence the Triangle inequality gives

$$|\zeta(2n+1) - (r - r')| = |t_n + \zeta(2n+1) - r - t_n + r'| < |t_n + \zeta(2n+1) - r| + |-t_n + r'| < \epsilon + \epsilon'$$

or better as $\forall \epsilon'' > 0\ \exists$ infinitely many $r'' \in \mathbb{Q}$ such that

$$\boxed{|\zeta(2n+1) - r''| < \epsilon''}$$

where $r'' = r - r'$ and $\epsilon'' = \epsilon + \epsilon'$ thus rendering $\zeta(2n+1)$ to be irrational. This completes the proof of our Theorem.

2.5. **What we reap of this.** We attain a bound for the value of the Riemann Zeta Function at Odd integers. One can see that

$$\boxed{\zeta(2n+1) < \frac{1}{2} + \frac{(2\pi)^{2n}}{4(2n)!}B_{2n}}$$

at $n = 1$ gives $\zeta(3) = 1.202056903$ and $\xi(n) = 1.3224670334$,both approximated.A more sharp bound may be obtained by other methods. But this bound approches 1 as $n \to \infty$(since $\zeta(2n)$ approaches 1).As $\zeta(2n+1) > 1$ and convergent(since the sequence $\frac{1}{k^{2n+1}}$ for a fixed natural number n is a monotonically decreasing sequence on $[1, 0)$) we see that Sandwich Theorem tells us $\lim_{n\to\infty} \zeta(2n+1) = 1$ which is at par with the established results.

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