ON A CONJECTURE OF ERDŐS ON ADDITIVE BASIS OF LARGE ORDERS

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Abstract. Using the methods of multivariate circles of partition, we prove that for any additive base $\mathbb{A}$ of order $h \geq 2$ the upper bound

$$\#\left\{(x_1, x_2, \ldots, x_h) \in \mathbb{A}^h \mid \sum_{i=1}^{h} x_i = k\right\} \ll_h \log k$$

holds for sufficiently large values of k provided the counting function

$$\#\left\{(x_1, x_2, \ldots, x_h) \in \mathbb{A}^h \mid \sum_{i=1}^{h} x_i = k\right\}$$

is an increasing function for all k sufficiently large.

1. Introduction

Let $\mathbb{A} \subset \mathbb{N}$ then we say $\mathbb{A}$ is an additive base of order h if the counting function

$$r_{\mathbb{A},h}(k) := \#\left\{(x_1, x_2, \ldots, x_h) \in \mathbb{A}^h \mid \sum_{i=1}^{h} x_i = k\right\} > 0$$

for all sufficiently large values of k. In [2], Erdős proved that there exists an additive base $\mathbb{A}$ of order 2 and some constant $c_1, c_2 > 0$ such that the inequalities

$$c_1 \log k \leq r_{\mathbb{A}}(n) \leq c_2 \log k$$

and conjectured that

$$\limsup_{k \longrightarrow \infty} \frac{r_{\mathbb{A}}(k)}{\log k} > 0$$

if $\mathbb{A}$ is an additive base of order $h \geq 2$.
In [1] we have developed a method for studying a large class of additive number theory problems. The method of circles of partition is very vast and rich and has yet unexplored applications. This method is easy to use given its combinatorial affinity. It is very elementary in nature and has parallels with configurations of points on the geometric circle.
It works in the following intuitive sense: Let us suppose that for any $n \in \mathbb{N}$ we can write $n = u + v$ where $u, v \in \mathbb{M} \subset \mathbb{N}$ then the new method associate each of this summands to points on the circle generated in a certain manner by $n > 2$ and a line joining any such associated points on the circle. This geometric correspondence turns out to useful in our development, as the results obtained in this setting are

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then transformed back to results concerning the partition of integers. We explore this method in the paper by introducing new notions to obtain the result

Theorem 1.1. *Let $\mathbb{A} \subset \mathbb{N}$ and $h \geq 2$ fixed. If*
$$\#\left\{\mathbb{L}_{[x_1],[x_2],\ldots,[x_h]} \hat{\in} \mathcal{C}(k, \bigotimes_{i=1}^{h}\mathbb{A})\right\} > 0$$
and $\#\left\{\mathbb{L}_{[x_1],[x_2],\ldots,[x_h]} \hat{\in} \mathcal{C}(k, \bigotimes_{i=1}^{h}\mathbb{A})\right\}$ is an increasing function for all sufficiently large values of k, then
$$\#\left\{\Re\mathbb{L}_{[x_1],[x_2],\ldots,[x_h]} \hat{\in} \mathcal{C}(k, \bigotimes_{i=1}^{h}\mathbb{A})\right\} \ll_h \log k$$
for all k sufficiently large.

This result is equivalent to the statement that
$$r_{\mathbb{A},h}(k) \ll_h \log k$$
for all sufficiently large values of k under the assumption that the counting function $r_{\mathbb{A},h}(k)$ is increasing for sufficiently large values of k.

In this paper we study a multivariate version of the method, where we allow our base regulators to be the direct product $\bigotimes$ of subsets of the natural numbers $\mathbb{N}$. With the goal of studying a general version of the Erdős-Turán additive base conjecture, we introduce and study the notion of the **axial** potential of the multivariate circle of partition.

Notations. We denote by $\mathbb{N}_n = \{m \in \mathbb{N} \mid \ m \leq n\}$ the **sequence** of the first n natural numbers

2. Multivariate circles of partition

In this section we introduce and study the notion of **multivariate** circles of partitions. We launch the following language.

Definition 2.1. Let $\mathbb{A} \subseteq \mathbb{N}$. Then we denote with
$$\mathcal{C}(n, \bigotimes_{i=1}^{h}\mathbb{A}_i) = \left\{[x_1], [x_2], \ldots, [x_h] \| \ x_i \in \mathbb{A}_i, \ n = \sum_{i=1}^{h} x_i\right\}$$
a **multivariate** circle of partition generated by $n \in \mathbb{N}$ with base **regulators** $\bigotimes_{i=1}^{h}\mathbb{A}_i$. We call members of the multivariate CoP multivariate points.

Definition 2.2. We denote the line $\mathbb{L}_{[x_1],[x_2],\ldots,[x_h]}$ joining the points $[x_1], [x_2], \ldots, [x_h]$ as a **axis** of the multivariate CoP $\mathcal{C}(n, \bigotimes_{i=1}^{h}\mathbb{A}_i)$ if and only if $x_i \in \mathbb{A}_i$ for each $1 \leq i \leq h$ and $n = \sum\limits_{i=1}^{h} x_i$. We say the axis points $[x_i]$ for each $1 \leq i \leq h$ are axis **residents**. We do not view the axis as any different among other axis $\mathbb{L}_{[x_1],[x_2],\ldots,[x_h]}$ up to the rearrangements of its residents points. In special cases where the points $[x_k] \in \mathcal{C}(n, \bigotimes_{i=1}^{h}\mathbb{A}_i)$ such that $hx_k = n$ then we call $[x_i]$ the **center** of the multivariate CoP. If it exists, then we call it as a **degenerated axis** $\mathbb{L}_{[x_k]}$

in comparison to the **real axes** $\mathbb{L}_{[x_1],[x_2],\ldots,[x_h]}$, where not all of the weights x_i can be equal. We denote the assignment of an axis $\mathbb{L}_{[x_1],[x_2],\ldots,[x_h]}$ to the multivariate CoP $\mathcal{C}(n,\bigotimes_{i=1}^{h}\mathbb{A}_i)$ as

$$\mathbb{L}_{[x_1],[x_2],\ldots,[x_h]} \hat{\in} \mathcal{C}(n,\bigotimes_{i=1}^{h}\mathbb{A}_i) \text{ which means } [x_1],[x_2],\ldots,[x_h] \in \mathcal{C}(n,\bigotimes_{i=1}^{h}\mathbb{A}_i)$$

with

$$n=\sum_{i=1}^{h}x_i$$

for a fixed $n\in\mathbb{N}$ with $x_i\in\mathbb{A}_i$ for each $1\leq i\leq h$ or vice versa and the number of real axes of the generalized CoP as

$$\nu(\mathcal{C}(n,\bigotimes_{i=1}^{h}\mathbb{A}_i) := \#\{\mathbb{L}_{[x_1],[x_2],\ldots,[x_h]} \hat{\in} \mathcal{C}(n,\bigotimes_{i=1}^{h}\mathbb{A}_i) \mid x_i\neq x_j\}.$$

for all $1<i<j\leq h$. The lines $\mathcal{L}_{[x_1],[x_2],\ldots,[x_h]}$ joining any h arbitrary points $[x_1],[x_2],\ldots,[x_h]\in\mathcal{C}(n,\bigotimes_{i=1}^{h}\mathbb{A}_i)$ which are not resident points in the multivariate CoP will be referred to as a **graph** induced by the multivariate CoP.

Throughout this paper we will denote for simplicity the multivariate circle of partition in simple wording as m-CoP. The notion of a multivariate axis is not technically convenient to work with; nonetheless, it is fairly manageable if we confine ourselves to a certain class of axis of a typical CoP. As it will prove very useful in the sequel and will feature very greatly in our results in the sequel, we find it more prudent exploiting the notion of representative axis.

Definition 2.3. Let $\mathcal{C}(n,\bigotimes_{i=1}^{h}\mathbb{A}_i)$ be a multivariate CoP and let $[x_1]\in\mathcal{C}(n,\bigotimes_{i=1}^{h}\mathbb{A}_i)$ be a fixed point. Then we say the axis $\mathbb{L}_{[x_1],[x_2],\ldots,[x_h]} \hat{\in} \mathcal{C}(n,\bigotimes_{i=1}^{h}\mathbb{A}_i)$ belongs to the class m axis of the multivariate CoP if

$$x_2+\cdots+x_h=m.$$

Proposition 2.4. *Let $\mathcal{C}(n,\bigotimes_{i=1}^{h}\mathbb{N})$ be a multivariate CoP. Then there are*

$$\left\lfloor\frac{n-1}{h}\right\rfloor$$

axis-classes of the multivariate CoP.

Throughout this paper we will work within the axis-classes and use their representatives. For any s axis-class of a multivariate CoP $\mathcal{C}(n,\bigotimes_{i=1}^{h}\mathbb{A}_i)$

$$\mathcal{C}_s := \{\mathbb{L}^{s}_{[x_1],[x_2],\ldots,[x_h]} \hat{\in} \mathcal{C}(n,\bigotimes_{i=1}^{h}\mathbb{A}_i)\}$$

we denote the representative axis of the class as $\Re(\mathcal{C}_s)$. Henceforth in counting the number of axis of a typical CoP we will only count the number of representative axis or simply the number of axis-classes. We denote more generally the set of all representative axis of the axis-classes as

$$\{\Re\mathbb{L}_{[x_1],[x_2],\ldots,[x_h]} \hat{\in} \mathcal{C}(n,\bigotimes_{i=1}^{h}\mathbb{A}_i)\}$$

and the number of all representative axis in the m-CoP as

$$\#\{\Re\mathbb{L}_{[x_1],[x_2],\ldots,[x_h]} \hat{\in} \mathcal{C}(n,\bigotimes_{i=1}^{h}\mathbb{A}_i)\}.$$

It has been observed that for a CoP with a natural number $\bigotimes_{i=1}^{h}\mathbb{N}$ base regulator the number of representative axis is basically the quantity

$$\left\lfloor \frac{n-1}{h} \right\rfloor.$$

3. Axial potential of multivariate circles of partition

In this section we introduce and study the notion of the **axial potential** of an m-CoP. We launch the following language.

Definition 3.1. Let $\mathcal{C}(n,\bigotimes_{i=1}^{h}\mathbb{A}_i)$ be an m-CoP. Then by the l^{th} **axial potential** denoted, $\lfloor \mathcal{C}(\infty,\bigotimes_{i=1}^{h}\mathbb{A}_i)\rfloor^l$, we mean the infinite sum

$$\lfloor \mathcal{C}(\infty,\bigotimes_{i=1}^{h}\mathbb{A}_i)\rfloor^l = \sum_{n=h+1}^{\infty} \frac{\#\left\{\Re\mathbb{L}_{[x_1],[x_2],\ldots,[x_h]} \hat{\in} \mathcal{C}(n,\bigotimes_{i=1}^{h}\mathbb{A}_i)\right\}^l}{\#\left\{\Re\mathbb{L}_{[x_1],[x_2],\ldots,[x_h]} \hat{\in} \mathcal{C}(n,\bigotimes_{i=1}^{h}\mathbb{A}_i \cup \bigotimes_{i=1}^{h}\mathbb{N})\right\}^l}.$$

We say the l^{th} axial potential is finite if the series converges; otherwise, we say it diverges.

Theorem 3.2 (Main theorem)**.** *Let $\mathbb{A}\subset\mathbb{N}$ and $h\geq 2$ fixed. If*

$$\#\left\{\mathbb{L}_{[x_1],[x_2],\ldots,[x_h]} \hat{\in} \mathcal{C}(k,\bigotimes_{i=1}^{h}\mathbb{A})\right\} > 0$$

and $\#\left\{\mathbb{L}_{[x_1],[x_2],\ldots,[x_h]} \hat{\in} \mathcal{C}(k,\bigotimes_{i=1}^{h}\mathbb{A})\right\}$ is an increasing function for all sufficiently large values of k, then

$$\#\left\{\Re\mathbb{L}_{[x_1],[x_2],\ldots,[x_h]} \hat{\in} \mathcal{C}(k,\bigotimes_{i=1}^{h}\mathbb{A})\right\} \ll_h \log k$$

for all k sufficiently large.

Proof. Under the requirement $\mathbb{A}\subset\mathbb{N}$ with $\#\left\{\mathbb{L}_{[x_1],[x_2],\ldots,[x_h]} \hat{\in} \mathcal{C}(k,\bigotimes_{i=1}^{h}\mathbb{A})\right\} > 0$ for all sufficiently large values of k, then it implies that

$$\#\left\{\Re\mathbb{L}_{[x_1],[x_2],\ldots,[x_h]} \hat{\in} \mathcal{C}(k,\bigotimes_{i=1}^{h}\mathbb{A})\right\} > 0$$

for all sufficiently large values of k so that under the assumption

$$\#\left\{\mathbb{L}_{[x_1],[x_2],\ldots,[x_h]} \hat{\in} \mathcal{C}(k,\bigotimes_{i=1}^{h}\mathbb{A})\right\}$$

is an increasing function for all sufficiently large values of k the inequality

$$\begin{aligned}\sum_{n=h+1}^{k}\frac{\#\left\{\Re\mathbb{L}_{[x_1],[x_2],\ldots,[x_h]} \hat{\in} \mathcal{C}(n,\bigotimes_{i=1}^{h}\mathbb{A})\right\}}{\#\left\{\Re\mathbb{L}_{[x_1],[x_2],\ldots,[x_h]} \hat{\in} \mathcal{C}(n,\bigotimes_{i=1}^{h}\mathbb{N})\right\}} &\leq \#\left\{\Re\mathbb{L}_{[x_1],[x_2],\ldots,[x_h]} \hat{\in} \mathcal{C}(k,\bigotimes_{i=1}^{h}\mathbb{A})\right\}\\ &\times \sum_{n=h+1}^{k}\frac{1}{\lfloor\frac{n-1}{h}\rfloor}\\ &\ll_h \#\left\{\Re\mathbb{L}_{[x_1],[x_2],\ldots,[x_h]} \hat{\in} \mathcal{C}(k,\bigotimes_{i=1}^{h}\mathbb{A})\right\}\\ &\times \sum_{n=h+1}^{k}\frac{1}{n}\\ &\ll_h \#\left\{\Re\mathbb{L}_{[x_1],[x_2],\ldots,[x_h]} \hat{\in} \mathcal{C}(k,\bigotimes_{i=1}^{h}\mathbb{A})\right\}\\ &\times \log k\end{aligned}$$

holds for all sufficiently large k for a fixed $h \geq 2$, since

$$\#\left\{\Re\mathbb{L}_{[x_1],[x_2],\ldots,[x_h]} \hat{\in} \mathcal{C}(n,\bigotimes_{i=1}^{h}\mathbb{N})\right\} = \lfloor\frac{n-1}{h}\rfloor.$$

On the other hand, we choose a constant $l := l(k) \geq 2$ such that

$$\frac{\#\left\{\Re\mathbb{L}_{[x_1],[x_2],\ldots,[x_h]} \hat{\in} \mathcal{C}(n,\bigotimes_{i=1}^{h}\mathbb{A}_i)\right\}}{\#\left\{\Re\mathbb{L}_{[x_1],[x_2],\ldots,[x_h]} \hat{\in} \mathcal{C}(n,\bigotimes_{i=1}^{h}\mathbb{N})\right\}} > \frac{\#\left\{\Re\mathbb{L}_{[x_1],[x_2],\ldots,[x_h]} \hat{\in} \mathcal{C}(k,\bigotimes_{i=1}^{h}\mathbb{A}_i)\right\}^2}{\#\left\{\Re\mathbb{L}_{[x_1],[x_2],\ldots,[x_h]} \hat{\in} \mathcal{C}(n,\bigotimes_{i=1}^{h}\mathbb{N})\right\}^l}$$

for all $n \geq h+1$, then the lower bound for the truncated 1^{st} axial potential satisfies

$$\begin{aligned}
\lfloor \mathcal{C}(k, \bigotimes_{i=1}^{h} \mathbb{A}_i) \rfloor &= \sum_{n=h+1}^{k} \frac{\#\left\{\Re\mathbb{L}_{[x_1],[x_2],\ldots,[x_h]} \hat{\in} \mathcal{C}(n, \bigotimes_{i=1}^{h} \mathbb{A}_i)\right\}}{\#\left\{\Re\mathbb{L}_{[x_1],[x_2],\ldots,[x_h]} \hat{\in} \mathcal{C}(n, \bigotimes_{i=1}^{h} \mathbb{N})\right\}} \\
&\geq \#\left\{\Re\mathbb{L}_{[x_1],[x_2],\ldots,[x_h]} \hat{\in} \mathcal{C}(k, \bigotimes_{i=1}^{h} \mathbb{A}_i)\right\}^2 \\
&\times \sum_{n=h+1}^{k} \frac{1}{\#\left\{\Re\mathbb{L}_{[x_1],[x_2],\ldots,[x_h]} \hat{\in} \mathcal{C}(n, \bigotimes_{i=1}^{h} \mathbb{N})\right\}^l} \\
&= \#\left\{\Re\mathbb{L}_{[x_1],[x_2],\ldots,[x_h]} \hat{\in} \mathcal{C}(k, \bigotimes_{i=1}^{h} \mathbb{A}_i)\right\}^2 \times \sum_{n=h+1}^{k} \frac{1}{\lfloor \frac{n-1}{h} \rfloor^l} \\
&= \#\left\{\Re\mathbb{L}_{[x_1],[x_2],\ldots,[x_h]} \hat{\in} \mathcal{C}(k, \bigotimes_{i=1}^{h} \mathbb{A}_i)\right\}^2 \times \sum_{m=1}^{\lfloor \frac{k-1}{h} \rfloor} \frac{1}{m^l} \\
&\gg_h \#\left\{\Re\mathbb{L}_{[x_1],[x_2],\ldots,[x_h]} \hat{\in} \mathcal{C}(k, \bigotimes_{i=1}^{h} \mathbb{A}_i)\right\}^2 \times \sum_{m=1}^{k} \frac{1}{m^l}
\end{aligned}$$

since $\#\left\{\Re\mathbb{L}_{[x_1],[x_2],\ldots,[x_h]} \hat{\in} \mathcal{C}(n, \bigotimes_{i=1}^{h} \mathbb{N})\right\} = \lfloor \frac{n-1}{h} \rfloor$. By combining the upper bound for the truncated 1^{st} axial potential with the lower bound and using the requirement that $\mathbb{A} \subset \mathbb{N}$ with $\#\left\{\mathbb{L}_{[x_1],[x_2],\ldots,[x_h]} \hat{\in} \mathcal{C}(k, \bigotimes_{i=1}^{h} \mathbb{A})\right\} > 0$ for all sufficiently large values of k, we obtain (by cancellation)

$$\#\left\{\Re\mathbb{L}_{[x_1],[x_2],\ldots,[x_h]} \hat{\in} \mathcal{C}(k, \bigotimes_{i=1}^{h} \mathbb{A})\right\} \ll_h (\sum_{m=1}^{k} \frac{1}{m^l})^{-1} \log k \ll_h \log k$$

since

$$\sum_{m=1}^{k} \frac{1}{m^l} > 1$$

and

$$\sum_{m=1}^{\infty} \frac{1}{m^l} < \infty$$

for $l \geq 2$. □

References

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