Kochen-Specker theorem on binary logic

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Abstract

We discuss the experimentally accessible Kochen-Specker theorem on the basis of binary logic in terms of a finite precision measurement, that is, the first result is $1 - \epsilon_1$ and the second result is $-1 + \epsilon_2$. Here, $\epsilon_j, j = 1, 2$ represents a noise for the jth outcome. Further we violate a Kochen-Specker inequality by using a finite precision measurement. We hope gently our discussions could contribute for formulating the Kochen-Specker theorem into an experimentally accessible theory in terms of a finite precision measurement.

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I. INTRODUCTION

Einstein, Podolsky, and Rosen propose the incompleteness argument [1]. A hidden-variable interpretation of the quantum theory is a topic of research [2, 3]. The no-hidden-variable theorem of Kochen and Specker (the KS theorem) is famous [4]. It is begun to research the KS theorem by using inequalities (see Refs. [5–10]). Such inequalities for testing the KS theorem are useful for experimental investigations [11].

However, an experimental violation of such an inequality does not always imply that the experiment is performed by a finite precision measurement. In fact, we violate such an inequality by using a finite/infinite precision measurement. Thus, we need more discussion concerning the experimental accessible of the Kochen-Specker theorem in terms of a finite precision measurement.

Meyer discusses that a finite precision measurement nullifies the KS theorem [12]. Cabello discusses that a finite precision measurement does not nullify the KS theorem [7]. Barrett and Kent give an opinion for the debate [13]. Commutativity, comeasurability, and contextuality in the Kochen-Specker arguments are discussed by Hofer-Szabó [14]. Experimental approach to demonstrating contextuality for qudits is discussed by Sohbi, Ohana, Zaquine, Diamanti, and Markham [15]. Nagata and Nakamura propose [16] the measurement theory, in qubits handling, based on the truth values, i.e., the truth T (1) for true and the falsity F (0) for false. The results of measurements are either 0 or 1 in an ideal case. The Kochen-Specker theorem is certified by binary logic using the measurement theory based on the truth values in an ideal case [10].

In this paper, we discuss the experimentally accessible Kochen-Specker theorem on the basis of binary logic in terms of a finite precision measurement, that is, the first result is $1 - \epsilon_1$ and the second result is $-1 + \epsilon_2$. Here, $\epsilon_j, j = 1, 2$ represents a noise for the jth outcome. Further we violate a Kochen-Specker inequality by using a finite precision measurement. We hope gently our discussions could contribute for formulating the Kochen-Specker theorem into an experimentally accessible theory in terms of a finite precision measurement.

II. KOCHEN-SPECKER THEOREM IS CERTIFIED BY BINARY LOGIC IN A FINITE PRECISION MEASUREMENT

First, we want to certify the Kochen-Specker theorem by binary logic using the measurement theory based on the truth values in an ideal case. Therefore, we consider the measurement theory, in qubits handling, based on the truth values, i.e., the truth T (1) for true and the falsity F (0) for false. The results of measurements are either 0 or 1 in an ideal case.

We consider a value V which is the sum of two noisy data in a finite precision measurement experiment. Let us consider the problem in a noisy environment. We introduce the map $f(x) = (-1 + \epsilon_2)^x - (1 - x)\epsilon_1$ from $\{0, 1\}$ to $\{1 - \epsilon_1, -1 + \epsilon_2\}$, where $\epsilon_j, j = 1, 2$ represents a noise and $O(\epsilon_j) \ll 1, j = 1, 2$. Then the possible values of the measured results are mapped into either $1 - \epsilon_1$ or $-1 + \epsilon_2$ from either 1 or 0. Then, we apply the usual hidden-variable theoretical analysis in a finite precision measurement.

In what follows, we derive a quantum mechanical condition. We assume the number of $1 - \epsilon_1$ is equal to the number of $-1 + \epsilon_2$. If the number of trials is 2, then we have

$$V = (1 - \epsilon_1) + (-1 + \epsilon_2) = \epsilon_2 - \epsilon_1, \qquad (1)$$

where the value V is under an assumption that we do not assign a definite value into each of experimental data.

This is a quantum mechanical case. In this quantum mechanical case, we have

$$V \times V = (\epsilon_2 - \epsilon_1)^2. \tag{2}$$

On the other hand, we derive a hidden valuable theoretical condition. We assign a definite value into each of experimental data. In this case, we can consider all observables commute simultaneously [17]. And the sum rule is equivalent to the product rule.

We can depict the predetermined "hidden" results r_1 and r_2 as follows: $r_1 = 1 - \epsilon_1$ and $r_2 = -1 + \epsilon_2$. Let us write V as follows:

$$V = r_1 + r_2. \tag{3}$$

In the following, we evaluate a value $(V \times V)$ and derive a specific necessary condition under an assumption that we assign a definite value into each of experimental data.

We introduce an assumption that the sum rule and the product rule commute with each other [18, 19]. A supposition that the sum rule and the product rule commute with each other means a supposition that the two operations Addition and Multiplication commute with each other (see [20]). In other words, the operation Addition is equivalent to the operation Multiplication. We have

$$\begin{aligned}
& V \times V \\
& = (r_1 + r_2) \times (r_1 + r_2) \\
& = (r_1 \times r_1) + (r_1 \times r_2) + (r_2 \times r_1) + (r_2 \times r_2) \\
& = r_1^2 + (r_1 + r_2) + (r_2 + r_1) + r_2^2 \\
& = r_1^2 + (r_1 + r_1) + (r_2 + r_2) + r_2^2 \\
& = r_1^2 + (r_1 \times r_1) + (r_2 \times r_2) + r_2^2 \\
& = 2((r_1)^2 + (r_2)^2) \\
& = 2((1 - \epsilon_1)^2 + (-1 + \epsilon_2)^2).
\end{aligned} \tag{4}$$

Thus, we have the following assumption concerning the hidden-variable theoretic realism:

$$V \times V = 2((1 - \epsilon_1)^2 + (-1 + \epsilon_2)^2). \tag{5}$$

We cannot assign simultaneously the truth value "1" for the two assumptions (2) and (5). We derive the Kochen-Specker theorem in a finite precision measurement. Thus, we cannot assign a definite value into each of experimental data in a noisy environment. The Kochen-Specker theorem is certified by binary logic in terms of a finite precision measurement.

Generally Multiplication (the product rule) is completed by Addition (the sum rule). Therefore, we think that Addition (the sum rule) of the starting point may be superior to any other case.

III. USING INEQUALITIES AS TESTS FOR THE KS THEOREM FOR THREE-PARTICLE STATES IN A FINITE PRECISION MEASUREMENT

In this section, we give a violation of a KS inequality proposed in Ref. [9, 10] by using a finite precision measurement. We consider a three-particle uncorrelated state.

$$|+++\rangle. \tag{6}$$

The KS inequality [9, 10] is as follows:

$$-2 \leq \langle III \rangle + \langle IXX \rangle + \langle XIX \rangle + \langle XXI \rangle \leq 2, \tag{7}$$

where I is the identity observable and X is the x-component of Pauli observables. As a matter of fact, all results are plus one in an ideal case. Hence we have, in an ideal case,

$$\begin{aligned}
& \langle III \rangle = 1, \\
& \langle IXX \rangle = 1, \\
& \langle XIX \rangle = 1, \\
& \langle XXI \rangle = 1.
\end{aligned} \tag{8}$$

Thus, the KS inequality is violated

$$4 < 2. \tag{9}$$

Thus, we violate the KS inequality when using an infinite precision measurement.

On the other hand, we introduce a finite precision measurement in a noisy case. To simplify the argumentations, all results are $1 - \epsilon$, where ϵ represents a noise and $O(\epsilon) \ll 1$.

$$\begin{aligned}
& \langle III \rangle = 1 - \epsilon, \\
& \langle IXX \rangle = 1 - \epsilon, \\
& \langle XIX \rangle = 1 - \epsilon, \\
& \langle XXI \rangle = 1 - \epsilon.
\end{aligned} \tag{10}$$

Thus, the KS inequality is as follows:

$$-2 \leq 4 - 4\epsilon \leq 2. \tag{11}$$

Hence, we have a violation of the KS inequality when $0 < \epsilon < 1/2$. Thus, we violate the KS inequality when using a finite precision measurement.

IV. CONCLUSIONS

In conclusions, we have shown non-classicality of noisy data on the basis of binary logic in a finite precision measurement. We have used the measurement theory, in qubits handling, based on the binary logic, i.e., the truth T (1) for true and the falsity F (0) for false. The results of measurements have been either 1 or 0 in an ideal case. We have considered whether we can assign the predetermined "hidden" results to numbers $1 - \epsilon_1$ and $-1 + \epsilon_2$ as in results of measurements in a thought experiment. We have discussed the Kochen-Specker theorem on the basis of binary logic in a finite precision measurement, that is, the first result is $1 - \epsilon_1$ and the second result is $-1 + \epsilon_2$. Here, $\epsilon_j, j = 1, 2$ has represented a noise for the jth outcome. Further we have violated a Kochen-Specker inequality by using a finite precision measurement.

We have hoped gently our discussions could contribute for formulating the Kochen-Specker theorem into an experimentally accessible theory in terms of a finite precision measurement.

Generally Multiplication (the product rule) is completed by Addition (the sum rule). Therefore, we think that Addition (the sum rule) of the starting point may be superior to any other case.

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DECLARATIONS

Ethical Approval

We are in an applicable thought to Ethical Approval.

Competing interests

The authors state that there is no conflict of interest.

Authors' contributions

Koji Nagata, Do Ngoc Diep, and Tadao Nakamura wrote and read the manuscript.

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Availability of data and materials

Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

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