THE ASYMPTOTIC BINARY GOLDBACH AND LEMOINE CONJECTURES

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Abstract. In this paper we use the former of the authors developed theory of **circles of partition** to investigate possibilities to prove the binary Goldbach as well as the Lemoine conjecture. We state the *squeeze principle* and its consequences if the set of all odd prime numbers is the base set. With this tool we can prove asymptotic versions of the binary Goldbach as well as the Lemoine conjecture.

1. **Introduction and Preliminaries**

The Goldbach conjecture was born in 1742 through a correspondence between the German mathematician Christian Goldbach and the Swiss mathematician Leonard Euler. There are two known versions of the problem: the binary case and the ternary situation. The binary version ask whether every even number greater than 6 can be represented as the sum of two primes, whereas the ternary version ask whether every odd number greater than 7 can be expressed as the sum of three primes. The ternary version, however, was very recently solved in the preprint [9] that compiled and build on several chain of works. Although the binary problem has not been solved yet, significant strides have been made on its variations. The first significant step in this direction can be found in (see [5]), which demonstrates that every even number can be expressed as the sum of at most C primes, where C is a practically computable constant. In the early twentieth century, G.H Hardy and J.E Littlewood assuming the Generalized Riemann hypothesis (see [2]), showed that the number of even numbers $\leq X$ and violating the binary Goldbach conjecture is much less than $X^{\frac{1}{2}+c}$, where c is a small positive constant. Using sieve theory techniques, Jing-run Chen [7] showed that every even number can either be written as a sum of two prime numbers or a prime number and a number which is a product of two primes. It is well known that almost all even numbers can be expressed as the sum of two prime numbers, with the density of even numbers representable in this fashion being one [4], [3]. It is also known that there exists a constant K such that any even number can be expressed as the sum of two prime numbers and a maximum of K powers of two, where $K = 13$ [8].

Lemoine's conjecture, on the other hand, is the assertion that every odd number greater than 5 can be written as the sum of a prime number and a double of a prime number. More formally, the conjecture states

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Conjecture 1.1. *The equation*

$$2n+1=p+2q$$

always has a solution in the primes (not necessarily distinct) for all $n>2$.

The conjecture was first posed by Émile Lemoine [1] in 1895 but was wrongly attributed to Hyman Levy [6], who had thought very deeply about it; hence, the name Lemoine or sometimes Levy conjecture. The conjecture is on par with other additive prime number problems like the binary Goldbach conjecture (see [3],[4],[7]) and the ternary Goldbach conjecture (see [9]). It is easy to see that the Lemoine conjecture is much stronger than and implies the ternary Goldbach conjecture.

We devised a method that we believe could be a useful tool and a recipe for analyzing issues pertaining to the partition of numbers in designated subsets of $\mathbb{N}$ in our work [10], which was partially inspired by the binary Goldbach conjecture and its variants. The technique is fairly simple, and it is similar to how the points on a geometric circle can be arranged.

In an effort to make our work more self-explanatory, we have chosen to provide a little background of the method of circles of partition from [10] in the following sequel.

Definition 1.2. Let $n\in\mathbb{N}$ and $\mathbb{M}\subseteq\mathbb{N}$. We denote with

$$\mathcal{C}(n,\mathbb{M})=\{[x]\mid x,y\in\mathbb{M},n=x+y\}$$

the **Circle of Partition** generated by n with respect to the subset $\mathbb{M}$. We will abbreviate this in the further text as CoP. We call members of $\mathcal{C}(n,\mathbb{M})$ as points and denote them by $[x]$. For the special case $\mathbb{M}=\mathbb{N}$ we denote the CoP shortly as $\mathcal{C}(n)$. We denote with $\|[x]\|:=x$ the **weight** of the point $[x]$ and correspondingly the weight set of points in the CoP $\mathcal{C}(n,\mathbb{M})$ as $\|\mathcal{C}(n,\mathbb{M})\|$. Obviously holds

$$\|\mathcal{C}(n)\|=\{1,2,\dots,n-1\}.$$

Definition 1.3. We denote the line $\mathbb{L}_{[x],[y]}$ joining the point $[x]$ and $[y]$ as an axis of the CoP $\mathcal{C}(n,\mathbb{M})$ if and only if $x+y=n$. We say the axis point $[y]$ is an axis partner of the axis point $[x]$ and vice versa. We do not distinguish between $\mathbb{L}_{[x],[y]}$ and $\mathbb{L}_{[y],[x]}$, since it is essentially the the same axis. The point $[x]\in\mathcal{C}(n,\mathbb{M})$ such that $2x=n$ is the **center** of the CoP. If it exists then we call it as a **degenerated axis** $\mathbb{L}_{[x]}$ in comparison to the **real axes** $\mathbb{L}_{[x],[y]}$. We denote the assignment of an axis $\mathbb{L}_{[x],[y]}$ to a CoP $\mathcal{C}(n,\mathbb{M})$ as

$$\mathbb{L}_{[x],[y]}\ \hat{\in}\ \mathcal{C}(n,\mathbb{M}),\text{ which means }[x],[y]\in\mathcal{C}(n,\mathbb{M})\text{ with }x+y=n.$$

Important properties of CoPs are

- Each axis is uniquely determined by points $[x]\in\mathcal{C}(n,\mathbb{M})$.
- Each point of a CoP $\mathcal{C}(n,\mathbb{M})$ except its center has exactly one axis partner.

We denote the assignment of an axis $\mathbb{L}_{[x],[y]}$ resp. $\mathbb{L}_{[x]}$ to a CoP $\mathcal{C}(n,\mathbb{M})$ as

$$\mathbb{L}_{[x],[y]}\ \hat{\in}\ \mathcal{C}(n,\mathbb{M}),\text{ which means }[x],[y]\in\mathcal{C}(n,\mathbb{M})\text{ and }x+y=n\text{ resp.}$$

$$\mathbb{L}_{[x]}\ \hat{\in}\ \mathcal{C}(n,\mathbb{M}),\text{ which means }[x]\in\mathcal{C}(n,\mathbb{M})\text{ and }2x=n$$

and the number of real axes of a CoP as

$$\nu(n,\mathbb{M}) := \#\{\mathbb{L}_{[x],[y]} \mathrel{\hat{\in}} \mathcal{C}(n,\mathbb{M}) \mid x < y\}. \tag{1.1}$$

Obviously holds

$$\nu(n,\mathbb{M}) = \left\lfloor \frac{k}{2} \right\rfloor, \text{ if } |\mathcal{C}(n,\mathbb{M})| = k.$$

2. The Squeeze Principle

In this section we introduce the *squeeze principle* and its consequences if the set of all odd prime numbers is the base set of CoPs.

Theorem 2.1 (The squeeze principle). *Let $\mathbb{B} \subset \mathbb{M} \subseteq \mathbb{N}$ and $\mathcal{C}(m,\mathbb{B})$ and $\mathcal{C}(m+t,\mathbb{B}) \neq \emptyset$ for $t \geq 4$. If there exists $\mathbb{L}_{[x],[y]} \mathrel{\hat{\in}} \mathcal{C}(m+t,\mathbb{M})$ with $x \in \mathbb{B}$ and $x < y$ such that*

$$y > w = \max\{u \in ||\mathcal{C}(m,\mathbb{M})|| \mid u \in \mathbb{B}\} > m - x, \tag{2.1}$$

then there exists $\mathcal{C}(s,\mathbb{B}) \neq \emptyset$ such that $m < s < m + t$.

Proof. In virtue of (2.1) holds $w \in \mathbb{B}$. As required the axis $\mathbb{L}_{[x],[y]} \mathrel{\hat{\in}} \mathcal{C}(m+t,\mathbb{M})$ exists with $x \in \mathbb{B}$ such that $m - w < x < y$. Then we have the inequality

$$\begin{aligned} m = w + (m - w) < \underline{w + x} &= w + (m + t - y) = m + t + (w - y) \\ &< m + t, \text{ since } y > w. \end{aligned} \tag{2.2}$$

Additional since $m - w < x = m + t - y$ holds $y - w < t$. With $s = \underline{w + x}$ there is an axis $\mathbb{L}_{[x],[w]} \mathrel{\hat{\in}} \mathcal{C}(s,\mathbb{B})$ because x as well as w are members of $\mathbb{B}$. It follows that $\mathcal{C}(s,\mathbb{B}) \neq \emptyset$ with $m < s < m + t$. □

Theorem 2.1 can be viewed as a basic tool-box for studying the possibility of partitioning numbers of a particular parity with components belonging to a special subset of the integers. It works by choosing two non-empty CoPs with the same base set and finding further non-empty CoPs with generators trapped in between these two generators. This principle can be used in an ingenious manner to study the broader question concerning the feasibility of partitioning numbers with each summand belonging to the same subset of the positive integers. We launch the following proposition as an outgrowth of Theorem 2.1.

Proposition 2.2 (The interval binary Goldbach partition detector). *Let $\mathbb{P}$ be the set of all prime numbers and $\mathcal{C}(m,\mathbb{P}), \mathcal{C}(m+t,\mathbb{P}) \neq \emptyset$ by $t \geq 4$. If there exists $\mathbb{L}_{[x],[y]} \mathrel{\hat{\in}} \mathcal{C}(m+t,\mathbb{N})$ with $x \in \mathbb{P}$ and $x < y$ such that*

$$y > w = \max\{u \in ||\mathcal{C}(m,\mathbb{N})|| \mid u \in \mathbb{P}\} > m - x \tag{2.3}$$

then there must exists $m < s < m + t$ such that $\mathcal{C}(s,\mathbb{P}) \neq \emptyset$.

Proof. This is a consequence of Theorem 2.1 by taking $\mathbb{M} = \mathbb{N}$ and $\mathbb{B} = \mathbb{P}$. □

Proposition 2.3 (Interval Goldbach partition). *Let $\mathbb{P}$ be the set of all prime numbers and $\mathcal{C}(m,\mathbb{P}), \mathcal{C}(m+t,\mathbb{P}) \neq \emptyset$ for $t \geq 4$. If $m - 1 \in \mathbb{P}$ then there exist some $s \equiv 0 (\bmod 2)$ with $m < s < m + t$ such that $\mathcal{C}(s,\mathbb{P}) \neq \emptyset$.*

Proof. Under the requirements $\mathcal{C}(m,\mathbb{P}),\mathcal{C}(m+t,\mathbb{P})\neq\emptyset$ for $t\geq 4$ and with w in virtue of (2.3), we choose $\mathbb{L}_{[3],[y]}\hat{\in}\mathcal{C}(m+t,\mathbb{N})$ so that $w=m-1$ and $y>w$ since $y=m+t-3>m$ for $t\geq 4$ and $m-1\in\mathbb{P}$. The inequality holds

$$y-w=y-(m-1)\leq(m+t-3)-(m-1)<t$$

and the conditions in Proposition 2.2 are satisfied, so that there exists some $s\equiv 0(\mathrm{mod}\,2)$ with $m<s<m+t$ such that $\mathcal{C}(s,\mathbb{P})\neq\emptyset$, f.i. $s=3+m-1=m+2$ with $\mathbb{L}_{[3],[m-1]}\hat{\in}\mathcal{C}(m+2,\mathbb{P})$. □

Proposition 2.4. *Let $\mathbb{P}$ be the set of all prime numbers and $\mathcal{C}(m,\mathbb{P}),\mathcal{C}(m+t,\mathbb{P})\neq\emptyset$ for $t\geq 4$ such that $m-1\in\mathbb{P}$. Then there are finitely many $s\equiv 0\pmod 2$ with $m<s<m+t$ such that $\mathcal{C}(s,\mathbb{P})\neq\emptyset$.*

Proof. The result is obtained by iterating repeatedly on the generators $s\equiv 0\pmod 2$ with $m<s<m+t$ such that $\mathcal{C}(s,\mathbb{P})\neq\emptyset$. □

Theorem 2.5 (Conditional Goldbach)**.** *Let $\mathbb{P}$ be the set of all prime numbers and $m\in 2\mathbb{N}$ such that $\mathcal{C}(m,\mathbb{P})\neq\emptyset$ for m **sufficiently** large. If for **all** $t\geq 4$ there exists $\mathbb{L}_{[x],[y]}\hat{\in}\mathcal{C}(m+t,\mathbb{N})$ with $x\in\mathbb{P}$ and $x<y$ such that*

$$y>w=\max\{u\in||\mathcal{C}(m,\mathbb{N})||\mid u\in\mathbb{P}\}>m-x,$$

then there are CoPs $\mathcal{C}(s,\mathbb{P})\neq\emptyset$ for all (sufficiently large) $s\in 2\mathbb{N}\mid s>m$.

Proof. It is known that there are infinitely many even numbers that can be written as the sum of two primes, so that for $m\in 2\mathbb{N}$ **sufficiently** large with $\mathcal{C}(m,\mathbb{P})\neq\emptyset$ then $t\geq 4$ can be chosen **arbitrarily** large such that $\mathcal{C}(m+t,\mathbb{P})\neq\emptyset$. Under the requirements and appealing to Proposition 2.2 there must exist some $s\equiv 0\pmod 2$ with $m<s<m+t$ such that $\mathcal{C}(s,\mathbb{P})\neq\emptyset$. Now we continue our arguments on the intervals of generators $[m,s]$ and $[s,s+r]$. If there exist some $u,v\in 2\mathbb{N}$ such that $m<u<s$ and $s<v<s+r$, then we repeat the argument under the requirements (for arbitrary t) to deduce that $\mathcal{C}(u,\mathbb{P})\neq\emptyset$ and $\mathcal{C}(v,\mathbb{P})\neq\emptyset$. We can iterate the process repeatedly so long as there exists some even generators trapped in the following sub-intervals of generators $[m,u],[u,s],[s,v],[v,v+r]$ where $v+r=m+t$ for $t\geq 4$. Since t can be chosen arbitrarily so that $\mathcal{C}(m+t,\mathbb{P})\neq\emptyset$, the assertion follows immediately. □

Now we use the *squeeze principle* to solve the Lemoine conjecture in analogy to its using for the binary Goldbach conjecture above.

Proposition 2.6 (The first interval Lemoine partition detector)**.** *Let $\mathbb{P}$ and $2\mathbb{P}$ be the set of all prime numbers and their doubles, respectively, and $\mathcal{C}(m,\mathbb{P}\cup 2\mathbb{P}),\mathcal{C}(m+t,\mathbb{P}\cup 2\mathbb{P})\neq\emptyset$ by $t\geq 4$. If there exists $\mathbb{L}_{[x],[y]}\hat{\in}\mathcal{C}(m+t,\mathbb{N})$ with $x\in\mathbb{P}$ and $x<y$ such that*

$$y>w=\max\{u\in||\mathcal{C}(m,\mathbb{N})||\mid u\in\mathbb{P}\cup 2\mathbb{P}\}\in 2\mathbb{P}>m-x \tag{2.4}$$

then there must exist $m<s<m+t$ such that $\mathcal{C}(s,\mathbb{P}\cup 2\mathbb{P})\neq\emptyset$.

Proof. This is a consequence of Theorem 2.1 by taking $\mathbb{M}=\mathbb{N}$ and $\mathbb{B}=\mathbb{P}\cup 2\mathbb{P}$. □

Proposition 2.7 (The second interval Lemoine partition detector)**.** *Let* $\mathbb{P}$ *and* $2\mathbb{P}$ *be the set of all prime numbers and their doubles, respectively, and* $\mathcal{C}(m,\mathbb{P}\cup 2\mathbb{P}),\mathcal{C}(m+t,\mathbb{P}\cup 2\mathbb{P})\neq\emptyset$ *by* $t\geq 4$*. If there exists* $\mathbb{L}_{[x],[y]}\hat{\in}\mathcal{C}(m+t,\mathbb{N})$ *with* $x\in 2\mathbb{P}$ *and* $x<y$ *such that*

$$y>w=\max\{u\in||\mathcal{C}(m,\mathbb{N})||\mid u\in\mathbb{P}\cup 2\mathbb{P}\}\in\mathbb{P}>m-x \tag{2.5}$$

then there must exist $m<s<m+t$ *such that* $\mathcal{C}(s,\mathbb{P}\cup 2\mathbb{P})\neq\emptyset$*.*

Proof. The proof is the same as in Proposition 2.6. □

Theorem 2.8 (Conditional Lemoine)**.** *Let* $\mathbb{P}$ *and* $2\mathbb{P}$ *be the set of all prime numbers and their doubles, respectively, and* $m\in 2\mathbb{N}+1$ *such that* $\mathcal{C}(m,\mathbb{P})\neq\emptyset$ *for* m ***sufficiently*** *large. If for* ***all*** $t\geq 4$ *there exists* $\mathbb{L}_{[x],[y]}\hat{\in}\mathcal{C}(m+t,\mathbb{N})$ *with* $x\in\mathbb{P}$ *and* $x<y$ *such that*

$$y>w=\max\{u\in||\mathcal{C}(m,\mathbb{N})||\mid u\in\mathbb{P}\cup 2\mathbb{P}\}\in 2\mathbb{P}>m-x,$$

or there exists $\mathbb{L}_{[x],[y]}\hat{\in}\mathcal{C}(m+t,\mathbb{N})$ *with* $x\in 2\mathbb{P}$ *and* $x<y$ *such that*

$$y>w=\max\{u\in||\mathcal{C}(m,\mathbb{N})||\mid u\in\mathbb{P}\cup 2\mathbb{P}\}\in\mathbb{P}>m-x$$

then there are CoPs $\mathcal{C}(s,\mathbb{P}\cup 2\mathbb{P})\neq\emptyset$ *for all (sufficiently large)* $s\in 2\mathbb{N}+1\mid s>m$*.*

Proof. It is known that there are infinitely many odd numbers that can be written as the sum of a prime and a double of a prime, so that for $m\in 2\mathbb{N}+1$ **sufficiently** large with $\mathcal{C}(m,\mathbb{P}\cup 2\mathbb{P})\neq\emptyset$ then $t\geq 4$ can be chosen **arbitrarily** large such that $\mathcal{C}(m+t,\mathbb{P})\neq\emptyset$. Under the requirements and appealing to Proposition 2.6 and 2.7 there must exist some $s\equiv 1\pmod 2$ with $m<s<m+t$ such that $\mathcal{C}(s,\mathbb{P}\cup 2\mathbb{P})\neq\emptyset$. Now we continue our arguments on the intervals of generators $[m,s]$ and $[s,s+r]$. If there exist some $u,v\in 2\mathbb{N}+1$ such that $m<u<s$ and $s<v<s+r$, then we repeat the argument under the requirements (for arbitrary t) to deduce that $\mathcal{C}(u,\mathbb{P}\cup 2\mathbb{P})\neq\emptyset$ and $\mathcal{C}(v,\mathbb{P}\cup 2\mathbb{P})\neq\emptyset$. We can iterate the process repeatedly so long as there exists some odd generators trapped in the following sub-intervals of generators $[m,u],[u,s],[s,v],[v,v+r]$ where $v+r=m+t$ for $t\geq 4$. Since t can be chosen arbitrarily so that $\mathcal{C}(m+t,\mathbb{P}\cup 2\mathbb{P})\neq\emptyset$, the assertion follows immediately. □

3. **Application to the Binary Goldbach Conjecture**

In this section we apply the notion of the quotient complex circles of partition and the squeeze principle to study the binary Goldbach conjecture in the very large. Despite Estermann's proof from 1938 (see [3]) that the binary Goldbach conjecture is true for almost all positive integers, we can use our tool to establish and prove independently the binary Goldbach conjecture in an asymptotic sense. We lay down the following elementary results which will feature prominently in our arguments.

Lemma 3.1 (The prime number theorem)**.** *Let* $\pi(m)$ *denotes the number of prime numbers less than or equal to* m *and* $p_{\pi(m)}$ *denotes the* $\pi(m)^{th}$ *prime number. Then we have the asymptotic relation*

$$p_{\pi(m)}\sim m\left(1-\frac{\log\log m}{\log m}\right).$$

Proof. This is an easy consequence by combining the two versions of the prime number theorem

$$\pi(m) \sim \frac{m}{\log m} \quad \textbf{and} \quad p_k \sim k \log k$$

where p_k denotes the k^{th} prime number. Since with $k = \pi(m)$ we get

$$\begin{aligned} p_k = p_{\pi(m)} &\sim \frac{m}{\log m} \log\left(\frac{m}{\log m}\right) \\ &= \frac{m}{\log m}(\log m - \log\log m) \\ &= m\left(1 - \frac{\log\log m}{\log m}\right). \end{aligned}$$

□

Obviously holds with the variable denotations from the previous section

$$w = \max\{u \in ||\mathcal{C}(m, \mathbb{N})|| \mid u \in \mathbb{P}\} = p_{\pi(m)}. \tag{3.1}$$

Lemma 3.2 (Bertrand's postulate)**.** *There exists a prime number in the interval* $(k, 2k)$ *for all* $k > 1$.

The formula in Lemma 3.1 obviously suggests that the $\pi(m)^{th}$ prime number satisfies and implies the asymptotic relation $p_{\pi(m)} \sim m$. While this is valid in practice, it does not actually help in measuring the asymptotic of the discrepancy between the maximum prime number less than m and m. It gives the misleading impression that this discrepancy has absolute difference tending to zero in the very large. We reconcile this potentially nudging flaw by doing things slightly differently.

Lemma 3.3 (The little lemma)**.** *Let* $\mathbb{P}$ *be the set of all prime numbers and* $m \in \mathbb{N}$ *be* ***sufficiently*** *large such that* $\mathcal{C}(m, \mathbb{P}) \neq \emptyset$. *Then for all* $x \in \mathbb{P}$ *satisfying* $\frac{m \log\log m}{\log m} < x < \frac{m \log(\log m)^2}{\log m}$ *the asymptotic relation and inequalities*

$$m - w \sim \frac{m \log\log m}{\log m}$$

and

$$0 \lesssim |w - (m + t - x)| \lesssim t$$

hold for $t \geq 4$.

Proof. Appealing to the prime number theorem, we obtain with (3.1) the asymptotic inequalities

$$\begin{aligned} m - w &= m - p_{\pi(m)} \\ &\sim m - m\left(1 - \frac{\log\log m}{\log m}\right) \\ &= \frac{m \log\log m}{\log m} \end{aligned}$$

for all sufficiently large $m \in 2\mathbb{N}$ and

$$\begin{aligned} m+t-x &> m+t-\frac{m\log(\log m)^2}{\log m} \\ &= m(1-\frac{\log(\log m)^2}{\log m})+t \\ &\sim m+t > p_{\pi(m)} = w \end{aligned}$$

and

$$\begin{aligned} |w-(m+t-x)| &= |m+t-x-p_{\pi(m)}| \\ &< |m+t-\frac{m\log\log m}{\log m}-p_{\pi(m)}| \\ &\sim |m+t-\frac{m\log\log m}{\log m}-m(1-\frac{\log\log m}{\log m})| \\ &= t \end{aligned}$$

for $t \geq 4$. $\square$

We are now ready to prove the binary Goldbach conjecture for all **sufficiently** large even numbers. The following result is a culmination and - to a larger extent - a mishmash of ideas espoused in this paper.

Theorem 3.4 (Asymptotic Goldbach theorem)**.** *Every* ***sufficiently*** *large even number can be written as the sum of two prime numbers.*

Proof. The claim is equivalent to the statement:

For every sufficiently large even number n holds $\mathcal{C}(n, \mathbb{P}) \neq \emptyset$.

It is known that there are infinitely many even numbers $m > 0$ with $\mathcal{C}(m, \mathbb{P}) \neq \emptyset$. Let us choose $m \in 2\mathbb{N}$ **sufficiently** large such that $\mathcal{C}(m, \mathbb{P}) \neq \emptyset$ and choose $t \geq 4$ such that $\mathcal{C}(m+t, \mathbb{P}) \neq \emptyset$. Let us choose a prime number $x < \frac{m\log(\log m)^2}{\log m}$ such that $x > \frac{m\log\log m}{\log m}$, since by Bertrand's postulate (Lemma 3.2) there exists a prime number x such that $x \in (k, 2k)$ for every $k > 1$. Then we get for the axis partner $[y]$ of the axis point $[x]$ of $\mathbb{L}_{[x],[y]} \hat{\in} \mathcal{C}(m+t, \mathbb{N})$ the inequality

$$\begin{aligned} y = m+t-x &> m+t-\frac{m\log(\log m)^2}{\log m} \\ &= m(1-\frac{\log(\log m)^2}{\log m})+t \\ &\sim m+t > p_{\pi(m)} = w \end{aligned}$$

for $t \geq 4$ and by appealing to Lemma 3.3 also the following asymptotic inequalities

$$m-w \sim \frac{m\log\log m}{\log m} < x$$

and

$$|y-w| = |(m+t-x)-w| = |m-w+t-x| \lesssim |x+t-x| = t.$$

Then the requirements in Theorem 2.5 are fulfilled **asymptotically** with

$$y \gtrsim w \text{ and } x \gtrsim m-w \text{ and } 0 \lesssim |y-w| \lesssim t$$

and the result follows by arbitrarily choosing $t \geq 4$ so that $\mathcal{C}(m+t, \mathbb{P}) \neq \emptyset$ and adapting the proof in Theorem 2.5. $\square$

4. **Application to the Lemoine Conjecture**

In this section we apply the notion of the quotient complex circles of partition and the squeeze principle to study Lemoine's conjecture in the very large. We begin with the following preparatory elementary results.

Lemma 4.1. *Let $\pi(m)$ denotes the number of prime numbers less than or equal to m and $p_{\pi(m)}$ denotes the $\pi(m)^{th}$ prime number. Then we have the asymptotic relations*

$$p_{\pi(m)} \sim m\left(1 - \frac{\log\log m}{\log m}\right) \sim 2p_{\pi(\frac{m}{2})}.$$

Proof. The left asymptotic relation is proved in Lemma 3.1. Now we replace in its result m by $\frac{m}{2}$ and get

$$\begin{aligned}2p_{\pi(\frac{m}{2})} &\sim 2\frac{m}{2}\left(1 - \frac{\log\log\frac{m}{2}}{\log\frac{m}{2}}\right)\\ &= m\left(1 - \frac{\log(\log m - \log 2)}{\log(\log m - \log 2)}\right)\\ &\sim m\left(1 - \frac{\log\log m}{\log m}\right).\end{aligned}$$

$\square$

Obviously holds with the variable denotations from the previous section

$$w = \max\{u \in ||\mathcal{C}(m,\mathbb{N})|| \ \mid u \in \mathbb{P} \cup 2\mathbb{P}\} = p_{\pi(m)} \tag{4.1}$$

provides $w \in \mathbb{P}$ and

$$w' = \max\{u \in ||\mathcal{C}(m,\mathbb{N})|| \ \mid u \in \mathbb{P} \cup 2\mathbb{P}\} = 2p_{\pi(\frac{m}{2})} \tag{4.2}$$

provides $w' \in 2\mathbb{P}$.

Lemma 4.2 (The first little lemma)**.** *Let $\mathbb{P}$ and $2\mathbb{P}$ be the set of all prime numbers and their doubles, respectively, and $m \in \mathbb{N}$ be **sufficiently** large such that $\mathcal{C}(m, \mathbb{P} \cup 2\mathbb{P}) \neq \emptyset$. Then for all $x' \in 2\mathbb{P}$ with $x' = 2x$ for $x \in \mathbb{P}$ satisfying*

$$\frac{m\log\log m}{2\log m} < x < \frac{m\log\log m}{\log m}$$

the asymptotic relation and inequalities

$$m - w \sim \frac{m\log\log m}{\log m}$$

and

$$0 \lesssim |w - (m + t - x')| \lesssim t$$

hold for $t \geq 4$.

Proof. Due to Lemma 3.2 there is a prime between $\frac{m\log\log m}{2\log m}$ and $\frac{m\log\log m}{\log m}$. Appealing to the prime number theorem, we obtain with (4.1) the asymptotic inequalities

$$\begin{aligned} m - w &= m - p_{\pi(m)} \\ &\sim m - m\left(1 - \frac{\log\log m}{\log m}\right) \\ &= \frac{m\log\log m}{\log m} \end{aligned}$$

for all sufficiently large $m \in 2\mathbb{N}+1$ and

$$\begin{aligned} m + t - x' = m + t - 2x &> m + t - \frac{2m\log\log m}{\log m} \\ &= m\left(1 - \frac{\log(\log m)^2}{\log m}\right) + t \\ &\sim m + t \geq p_{\pi(m)} = w \end{aligned}$$

and

$$\begin{aligned} |w - (m + t - x')| &= |m + t - x' - p_{\pi(m)}| \\ &< |m + t - \frac{m\log\log m}{\log m} - p_{\pi(m)}| \\ &\sim \left|m + t - \frac{m\log\log m}{\log m} - m\left(1 - \frac{\log\log m}{\log m}\right)\right| \\ &= t \end{aligned}$$

for $t \geq 4$. □

Lemma 4.3 (The second little lemma)**.** *Let $\mathbb{P}$ be the set of all prime numbers and their doubles, respectively, and $m \in \mathbb{N}$ be* ***sufficiently*** *large such that $\mathcal{C}(m, \mathbb{P}\cup 2\mathbb{P}) \neq \emptyset$. Then for all $x \in \mathbb{P}$ satisfying*

$$\frac{m\log\log m}{\log m} < x < \frac{m\log(\log m)^2}{\log m}$$

the asymptotic relation and inequalities

$$m - w' \sim \frac{m\log\log m}{\log m}$$

and

$$0 \lesssim |w' - (m + t - x)| \lesssim t$$

hold for $t \geq 4$.

Proof. Due to Lemma 3.2 there is a prime between $\frac{m\log\log m}{\log m}$ and $\frac{m\log(\log m)^2}{\log m}$. Appealing to the prime number theorem, we obtain with (4.2) the asymptotic inequalities

$$\begin{aligned} m - w' &= m - 2p_{\pi(\frac{m}{2})} \\ &\sim m - m\left(1 - \frac{\log\log m}{\log m}\right) \\ &= \frac{m\log\log m}{\log m} \end{aligned}$$

for all sufficiently large $m \in 2\mathbb{N}+1$ and

$$\begin{aligned} m + t - x &> m + t - \frac{m\log(\log m)^2}{\log m} \\ &= m\left(1 - \frac{\log(\log m)^2}{\log m}\right) + t \\ &\sim m + t \geq p_{\pi(m)} \sim 2p_{\pi(\frac{m}{2})} = w' \end{aligned}$$

and

$$\begin{aligned} |w' - (m+t-x)| &= |m + t - x - 2p_{\pi(\frac{m}{2})}| \\ &< |m + t - \frac{m\log\log m}{\log m} - 2p_{\pi(\frac{m}{2})}| \\ &\sim \left|m + t - \frac{m\log\log m}{\log m} - m\left(1 - \frac{\log\log m}{\log m}\right)\right| \\ &= t \end{aligned}$$

for $t \geq 4$. □

We are now ready to prove the Lemoine conjecture for all **sufficiently** large odd numbers. It is a case-by-case argument and a culmination of ideas espoused in this paper.

Theorem 4.4 (Asymptotic Lemoine theorem)**.** *Every **sufficiently** large odd number can be written as a sum of a prime number and a double of a prime number.*

Proof. The claim is equivalent to the statement:

For every sufficiently large odd number $n \in 2\mathbb{N}+1$ holds $\mathcal{C}(n, \mathbb{P} \cup 2\mathbb{P}) \neq \emptyset$ since only the sum of an odd and an even number provides an odd number and therefore each axis $\mathbb{L}_{[x],[y]} \hat{\in} \mathcal{C}(m, \mathbb{P} \cup 2\mathbb{P})$ has an odd and an even axis point.

It is known that there are infinitely many odd numbers $m > 0$ with $\mathcal{C}(m, \mathbb{P} \cup 2\mathbb{P}) \neq \emptyset$. Let us choose $m \in 2\mathbb{N}+1$ **sufficiently** large such that $\mathcal{C}(m, \mathbb{P} \cup 2\mathbb{P}) \neq \emptyset$ and choose $t \geq 4$ such that $\mathcal{C}(m+t, \mathbb{P} \cup 2\mathbb{P}) \neq \emptyset$. Now, we distinguish and examine two special cases as below:

- The case

$$w = \max\{u \in ||\mathcal{C}(m, \mathbb{N})|| \ \mid u \in \mathbb{P} \cup 2\mathbb{P}\} = p_{\pi(m)}$$

- The case

$$w' = \max\{u \in ||\mathcal{C}(m, \mathbb{N})|| \ \mid u \in \mathbb{P} \cup 2\mathbb{P}\} = 2p_{\pi(\frac{m}{2})}$$

In the case

$$w = \max\{u \in ||\mathcal{C}(m,\mathbb{N})|| \ \mid u \in \mathbb{P} \cup 2\mathbb{P}\} = p_{\pi(m)}$$

then we choose a prime number $x < \dfrac{m \log\log m}{\log m}$ such that $x > \dfrac{m \log\log m}{2\log m}$, since by Bertrand's postulate (Lemma 3.2) there exists a prime number x such that $x \in (k, 2k)$ for every $k > 1$ and set $2x = x' \in 2\mathbb{P}$. Then we get for the axis partner $[y]$ of the axis point $[x']$ of $\mathbb{L}_{[x'],[y']} \hat{\in} \mathcal{C}(m+t,\mathbb{N})$ the inequality

$$\begin{aligned} y' = m + t - x' &> m + t - \frac{m \log(\log m)^2}{\log m} \\ &= m\left(1 - \frac{\log(\log m)^2}{\log m}\right) + t \\ &\sim m + t \geq p_{\pi(m)} = w \end{aligned}$$

for $t \geq 4$ and by appealing to Lemma 4.2 also the following asymptotic inequalities

$$m - w \sim \frac{m \log\log m}{\log m} < x'$$

and

$$|y' - w'| = |(m + t - x') - w| = |m - w + t - x'| \lesssim |x' + t - x'| = t.$$

Then the requirements in Theorem 2.8 are fulfilled **asymptotically** in this case with

$$y' \gtrsim w \text{ and } x' \gtrsim m - w \text{ and } 0 \lesssim |y' - w| \lesssim t.$$

In the case

$$w' = \max\{u \in ||\mathcal{C}(m,\mathbb{N})|| \ \mid u \in \mathbb{P} \cup 2\mathbb{P}\} = 2p_{\pi(\frac{m}{2})}$$

then we choose a prime number $x < \dfrac{m \log(\log m)^2}{\log m}$ such that $x > \dfrac{m \log\log m}{\log m}$, since by Bertrand's postulate (Lemma 3.2) there exists a prime number x such that $x \in (k, 2k)$ for every $k > 1$. Then we get for the axis partner $[y]$ of the axis point $[x]$ of $\mathbb{L}_{[x],[y]} \hat{\in} \mathcal{C}(m+t,\mathbb{N})$ the inequality

$$\begin{aligned} y = m + t - x &> m + t - \frac{m \log(\log m)^2}{\log m} \\ &= m\left(1 - \frac{\log(\log m)^2}{\log m}\right) + t \\ &\sim m + t \geq p_{\pi(m)} \sim 2p_{\pi(\frac{m}{2})} = w' \end{aligned}$$

for $t \geq 4$ and by appealing to Lemma 4.3 also the following asymptotic inequalities

$$m - w' \sim \frac{m \log\log m}{\log m} < x$$

and

$$|y - w'| = |(m + t - x) - w'| = |m - w' + t - x| \lesssim |x + t - x| = t.$$

Then the requirements in Theorem 2.8 are fulfilled **asymptotically** in this second case with

$$y \gtrsim w' \text{ and } x \gtrsim m - w' \text{ and } 0 \lesssim |y - w'| \lesssim t.$$

The result follows by arbitrarily choosing $t \geq 4$ so that $\mathcal{C}(m+t, \mathbb{P} \cup 2\mathbb{P}) \neq \emptyset$ and adapting the proof in Theorem 2.8. $\square$

Theorem 3.4 as well as Theorem 4.4 is equivalent to the statement: there must exist some positive constant N such that for all $m \geq N$, then it is always possible to partition every odd number m as a sum of two primes resp. as a sum of a prime and a double of a prime. This result - albeit constructive to some extent - looses its constructive flavour so that we cannot carry out this construction to cover all odd numbers, since we are unable to obtain any quantitative (lower) bound for the threshold N. At least, we are able to get a handle on the conjecture **asymptotically**.

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