The Collatz conjecture is most likely not true

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Abstract

We show that the Collatz undirected graph which has all natural numbers as its nodes is disconnected.

Introduction

Define the iterating function introduced by R. Terras[1] :

$$a_{n+1} = (3^{b} a_n + b)/2 \qquad (1)$$

where $b = 1$ when a_n is odd and $b = 0$ when a_n is even. The Collatz conjecture asserts that by starting with any positive integer a_0, there exists a natural number k such that $a_k = 1$.

1. The Collatz directed graph

Denote each natural number as node in a graph G , by eq.(1) each node in G can have at most two incoming node, and can have at most one outgoing node as shown in Figure 1. This graph is called the Collatz directed graph.

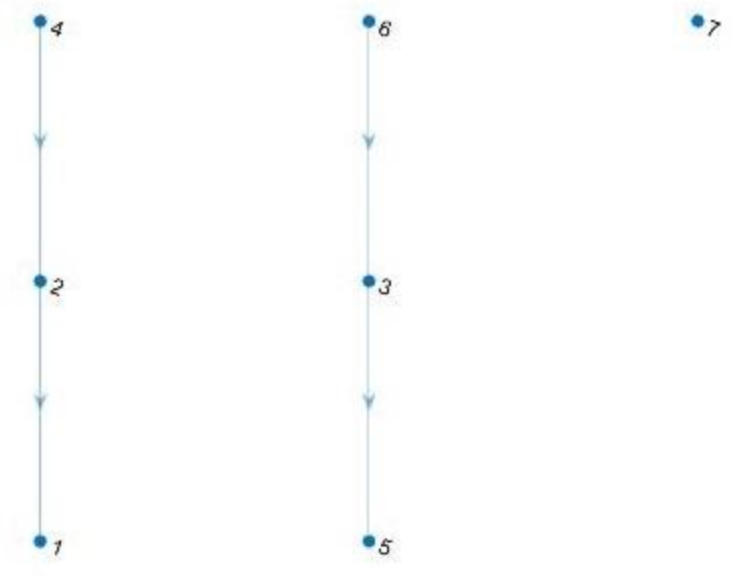

Figure 1. 7-nodes Collatz directed graph

2. The Collatz simple graph

The Collatz simple graph is obtained from the Collatz directed graph by changing it to the undirected graph as shown in Figure 2.

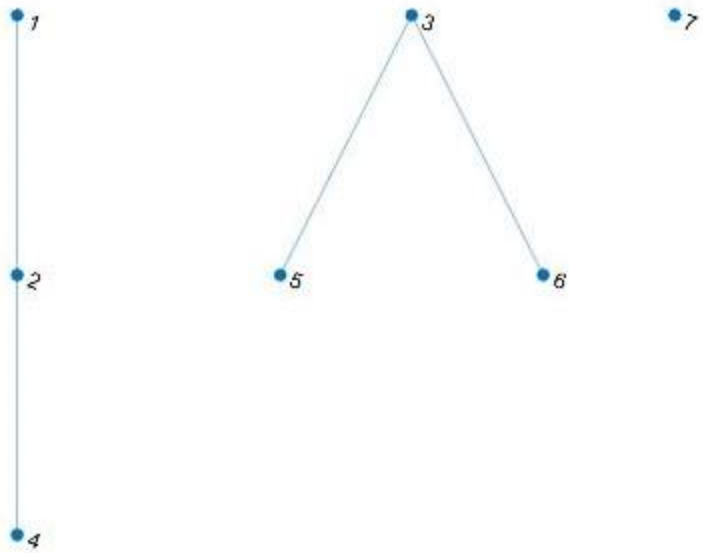

Figure 2. 7-nodes Collatz simple graph

The adjacency matrix A of the Collatz simple graph has its elements

$$A(2n,n)= A(n,2n) = 1;$$

$$A(2n+1,3n+2) = A(3n+2,2n+1) = 1, n=1,2,3\ldots\ldots$$

All other elements are zeros.

The adjacency matrix A of the 7-nodes Collatz simple graph is obtained as

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Note that this matrix is symmetric.

The Laplacian matrix is a matrix representation of a graph. It relates to many useful properties of a graph, such as its connectivity [2].

Given a simple graph G with nodes $v_1, \ldots\ldots, v_n$, its Laplacian matrix L_{nxn} is defined as

$$L_{nxn} = D - A,$$

where D is the degree matrix and A is the adjacency matrix of the graph G. The degree matrix is a diagonal matrix which has the number of edges attached to each node as its diagonal elements.

For a simple graph G with n nodes and its Laplacian matrix L with Eigenvalues $0 = \lambda_1 \leq \lambda_2 \leq \ldots\ldots \leq \lambda_n$ if $\lambda_2 > 0$ then a graph is fully connected. The Laplacian matrix of the 7-nodes Collatz simple graph is obtained as

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & -1 & -1 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Its eigenvalues are 0, 0, 0, 1, 1,3,3. Since $\lambda_2 = 0$ then this graph is disconnected.

3. The Infinite Collatz simple graph

For 2-nodes Collatz simple graph,

$$L_{2x2} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix},$$

Its eigenvalue are 0 and 2. Since $\lambda_2 > 0$ then this graph is connected.

For 3-nodes Collatz simple graph,

$$L_{3x3} = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Its eigenvalue are 0, 0 and 2. Since $\lambda_2 = 0$ then this graph is disconnected.

Theorem 1. If m-nodes Collatz simple graph is disconnected then (m+1)-nodes Collatz simple graph is also disconnected.

Proof. Let m-nodes Collatz simple graph consists of at least two isolated subgraphs. Consider two cases of m+1,

1. (m+1) is a positive odd integer and

   1.1 (m+1) ≡ 0 mod 3 or (m+1) ≡ 1 mod 3.

   In this case node (m+1) can connect to node 2(m+1) and node{3(m+1)+1}/2 but these node are outside of (m+1)-nodes graph. Therefore, node (m+1) is isolated.

   1.2 (m+1) ≡ 2 mod 3.

   In this case node (m+1) can connect to node {2(m+1)-1}/3 which is in one of subgraphs . Therefore, (m+1)-nodes Collatz simple graph will consist of at least two isolated subgraphs.

2. (m+1) is a positive even integer and

   2.1 (m+1) ≡ 0 mod 3 or (m+1) ≡ 1 mod 3.

   In this case node (m+1) can connect to node (m+1)/2 which is in one of subgraphs . Therefor, (m+1)-nodes Collatz simple graph will consist of at least two isolated subgraphs.

   2.2 (m+1) ≡ 2 mod 3.

   In this case node (m+1) can connect to node (m+1)/2 and node{2(m+1)-1}/3. We know that when (m+1) ≡ 2 mod 3 then m ≡ 1 mod 3. Therefore, (m+1)-nodes Collatz simple graph consist of at least one isolated node.

Since 3-nodes Collatz simple graph is disconnected then all n-nodes Collatz simple graph are disconnected for n = 4, 5, 6, ………∞.

4. Conclusion

The above analysis shows that an infinite Collatz simple graph is disconnected. Therefore, the Collatz conjecture is most likely not true.

References

[1] R . Terras, (1976). " A stopping time problem on the positive integers". Acta Arithmetica, 30(3), 241-252.

[2] D. A. Spielman, " The Laplacian ", Lecture 2, September 4, 2009. (https://www.cs.yale.edu/homes/spielman/561/2009/lect02-09.pdf).