A Problem on Sums of Powers

Suaib Lateef
Oke Oore, Iwo, Osun State, Nigeria.
lateefholuyinka@gmail.com

1 Problem Statement

The motivation for the problem to be presented is based on Newton's identity which states that

$$\sum_{i=k-n}^{k}(-1)^{i-1}e_{k-i}(x_1,\ldots,x_n)p_i(x_1,\ldots,x_n)=0,$$

for all $k>n\geq 1$, where $e_k(x_1,\ldots,x_n)$, for $k\geq 0$, is the sum of all distinct products of k distinct variables, and $p_k(x_1,\ldots,x_n)=\sum_{i=1}^{n}x_i^k$, where $k\geq 1$.

2 Problem

For $n\geq 3$, if

$$\sum_{k=1}^{n-1}x_k=y_1=2n-2,$$

$$\sum_{k=1}^{n-1}x_k^2=y_2=3n-2,$$

$$\sum_{k=1}^{n-1}x_k^3=y_3=4n-2,$$

$$\vdots$$

$$\sum_{k=1}^{n-1}x_k^{n-1}=y_{n-1}=n^2-2,$$

then

$$\sum_{k=1}^{n-1}x_k^{n}=y_n=n^2+n-2.$$

Essentially, $y_1,y_2,y_3\ldots y_n$ are in arithmetic progression(A.P) whose first term is $2n-2$ and common difference is n.

3 Special Cases

CASE n=3:

We see when $n = 3$ that

$$x_1 + x_2 = 4,$$

$$x_1^2 + x_2^2 = 7.$$

Now, we need to show that $x_1^3 + x_2^3 = 10$.
We know that

$$x_1^3 + x_2^3 = (x_1 + x_2)^3 - 3x_1x_2(x_1 + x_2),$$

$$(x_1^3 + x_2^3) = 64 - 12x_1x_2. \tag{1}$$

Also, we know that

$$(x_1 + x_2)^2 - (x_1^2 + x_2^2) = 2x_1x_2,$$

$$4^2 - 7 = 2x_1x_2,$$

$$x_1x_2 = \frac{9}{2}. \tag{2}$$

Putting (2) in (1), we see that

$$x_1^3 + x_2^3 = 10.$$

And indeed,

$$\left(\frac{4 + i\sqrt{2}}{2}\right) + \left(\frac{4 - i\sqrt{2}}{2}\right) = 4,$$

$$\left(\frac{4 + i\sqrt{2}}{2}\right)^2 + \left(\frac{4 - i\sqrt{2}}{2}\right)^2 = 7,$$

$$\left(\frac{4 + i\sqrt{2}}{2}\right)^3 + \left(\frac{4 - i\sqrt{2}}{2}\right)^3 = 10.$$

CASE n=4:

We see when $n = 4$ that

$$x_1 + x_2 + x_3 = 6,$$

$$x_1^2 + x_2^2 + x_3^2 = 10,$$

$$x_1^3 + x_2^3 + x_3^3 = 14.$$

Now, we need to show that $x_1^4 + x_2^4 + x_3^4 = 18$.
We know that

$$x_1^2 + x_2^2 + x_3^2 = (x_1 + x_2 + x_3)^2 - 2(x_1x_2 + x_2x_3 + x_1x_3),$$

$$10 = 36 - 2(x_1x_2 + x_2x_3 + x_1x_3),$$

$$x_1x_2 + x_2x_3 + x_1x_3 = 13,$$
$$(x_1x_2 + x_2x_3 + x_1x_3)^2 = 169,$$
$$x_1^2x_2^2 + x_2^2x_3^2 + x_1^2x_3^2 + 2x_1x_2x_3(x_1 + x_2 + x_3) = 169,$$
$$x_1^2x_2^2 + x_2^2x_3^2 + x_1^2x_3^2 + 12x_1x_2x_3 = 169,$$

$$\frac{(x_1^2 + x_2^2 + x_3^2)^2 - (x_1^4 + x_2^4 + x_3^4)}{2} + 12x_1x_2x_3 = 169. \tag{3}$$

Also, we know that

$$x_1^3 + x_2^3 + x_3^3 - 3x_1x_2x_3 = (x_1 + x_2 + x_3)(x_1^2 + x_2^2 + x_3^2 - x_1x_2 - x_2x_3 - x_1x_3),$$

we have

$$14 - 3x_1x_2x_3 = 6(10 - 13),$$

$$x_1x_2x_3 = \frac{32}{3}. \tag{4}$$

Putting (4) into (3), we have

$$\frac{10^2 - (x_1^4 + x_2^4 + x_3^4)}{2} + 12\left(\frac{32}{3}\right) = 169,$$

$$x_1^4 + x_2^4 + x_3^4 = 18.$$

And indeed, if

$$\alpha = \sqrt[3]{\frac{1}{3} + \frac{2\sqrt{3}}{9}} + \sqrt[3]{\frac{1}{3} - \frac{2\sqrt{3}}{9}} + 2,$$

then

$$\alpha + \left(\frac{-3\alpha(\alpha-6) + i\sqrt{384\alpha - (3\alpha(\alpha-6))^2}}{6\alpha}\right) + \left(\frac{-3\alpha(\alpha-6) - i\sqrt{384\alpha - (3\alpha(\alpha-6))^2}}{6\alpha}\right) = 6,$$

$$\alpha^2 + \left(\frac{-3\alpha(\alpha-6) + i\sqrt{384\alpha - (3\alpha(\alpha-6))^2}}{6\alpha}\right)^2 + \left(\frac{-3\alpha(\alpha-6) - i\sqrt{384\alpha - (3\alpha(\alpha-6))^2}}{6\alpha}\right)^2 = 10,$$

$$\alpha^3 + \left(\frac{-3\alpha(\alpha-6) + i\sqrt{384\alpha - (3\alpha(\alpha-6))^2}}{6\alpha}\right)^3 + \left(\frac{-3\alpha(\alpha-6) - i\sqrt{384\alpha - (3\alpha(\alpha-6))^2}}{6\alpha}\right)^3 = 14,$$

$$\alpha^4 + \left(\frac{-3\alpha(\alpha-6) + i\sqrt{384\alpha - (3\alpha(\alpha-6))^2}}{6\alpha}\right)^4 + \left(\frac{-3\alpha(\alpha-6) - i\sqrt{384\alpha - (3\alpha(\alpha-6))^2}}{6\alpha}\right)^4 = 18.$$

4 Conclusion

We have proven the special cases $n = 3$ and $n = 4$ of our claim. To prove higher cases of n using the method used here(which is inefficient to prove all the cases of n) is long and complicated. We hope a profound technique will be used to prove every integer n.

References

[1] 'Newton's identities' (2022) Wikipedia. Available at: `https://en.m.wikipedia.org/wiki/Newton%27s_identities` (Accessed: September 5, 2022)