LOCALLY UNIFORM APPROXIMATIONS AND RIEMANN HYPOTHESES (REVISED)

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To the memory of my dear parents
and my master Professor Jun-Ichi Igusa

Abstract. This paper offers a breakthrough in proving the veracity of original Riemann hypothesis, and extends the validity of its method to include the cases of the Dedekind zeta functions, the Hecke L-functions hence the Artin L-functions, and the Selberg class.

First we parametrize the Riemann surface **S** of log-function, with which we first shrink the scale of each chosen parameter for which it depends on the chosen natural number Q_{N_0} which is a chosen common multiple of all the denominators which are derived from a pre-set choice of rational numbers which approximate the values $\log(k+1)$ with the integers k in $0 \leq k \leq N$.

Then in (1.7) we define the mapping $-Q_{N_0}\log(.)$ to pull the truncated Dirichlet η-function $f_N(s)$ back to be re-defined on **S**, after that we shrink all the points to have their absolute values are all less than 1 and closer to 1. We apply the Euler transformation to the alternative series of Dirichlet η-functions $f(s)$ which are defined in (1.4), then we build up the locally uniform approximation of Theorem 4.7 for $f(s)$ which are established on any given compact subset contained in the right half complex plane.

In the second part we define the functions $\phi(s)$ which are formulated in (6.1) then by specific property of the functions $\phi(s)$, we have the similar asymptotics Theorem 6.5 as those of Theorem 4.7 to obtain the result of Theorem 6.8.

And with the locally uniform estimation Lemma 5.10, finally in Theorem 5.9 and Theorem 6.9 we employ Theorem 5.8 and Theorem 6.8 to solve problems of Riemann hypothesis for the Dedekind zeta functions, the Hecke L-functions, the Artin L-functions, and the Selberg class for which all of their nontrivial zeros are contained in the vertical line $Re(s) = 1/2$.

Finally for the $\gamma(s)$-factor of each Dirichlet series $D(s)$ which is formulated in (1.4), then by Theorem 6.9 it has neither zeros nor poles contained in the critical strip $0 < Re(s) < 1$ and the non-existence of Siegel's zeros for such Dirichlet series $D(s)$ is confirmed.

Contents

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1. INTRODUCTION

We recall the Riemann zeta function $\zeta(s)$ which is expressed in the following infinite sum

$$\zeta(s) = \frac{1}{1-2^{1-s}} \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \sum_{k=0}^{n} \binom{n}{k} (-1)^k (k+1)^{-s} \ , \ s \neq 1 \ , \tag{1.1}$$

and reset formula (1.1) by introducing the following Dirichlet η-function

$$f(s) := (1-2^{1-s})\zeta(s) \ . \tag{1.2}$$

Hence by the locally uniform convergence of the Dirichlet η-function $f(s)$ defined in (1.2), the Riemann zeta function $\zeta(s)$ defined in (1.1) converges locally uniformly on any compact subset contained in the punctured complex plane $\mathbb{C} - \{1\}$. This is proved by Helmut Hasse[HH30] and later by J. Sondow[JS94].

To prove it, they first define the finite differences

$$\Delta_n(\frac{1}{\nu^s}) := \sum_{k=0}^{n} \binom{n}{k} (-1)^k (k+1)^{-s} \qquad \text{for } n = 0, 1, 2, \ldots, \text{etc.},$$

and prove that for the Dirichlet η-function $f(s)$ defined in (1.2), it converges uniformly on any given compact subset contained in the entire complex plane $\mathbb{C}$.

Now to study the validity of the Riemann hypothesis in a general perspective, we introduce the following Dirichlet series $D(s)$

$$D(s) := \sum_{n=0}^{\infty} \frac{a_n}{n^s} \ , \quad Re(s) > 1 \ , \tag{1.3}$$

where all the a_n are the complex numbers such that for each sufficiently large n, each a_n is restricted to $a_n = o(n^{\varepsilon})$ for arbitrarily small $\varepsilon > 0$.

And for each given non-negative integer d_0 then by the general fact of the absolutely and locally uniform convergence

$$\begin{aligned} &(1-2^{1-s})^{d_0} \sum_{n=1}^{\infty} \frac{a_n}{n^s} \\ &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1} b_n}{n^s} \qquad b_n = o(n^{\varepsilon}) \ \text{for any } \varepsilon > 0 \\ &= \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \sum_{k=0}^{n} \binom{n}{k} (-1)^k b_{k+1} (k+1)^{-s} \ , \quad Re(s) > 1 \ , \end{aligned} \tag{1.4}$$

for which we will deal with them in the Key lemmas of Section 2, and by that we generalize the concept of the Dirichlet η-function $f(s)$ for which we will define it in Section 3 thus, with that we intend to study: (1) Analytic continuation of the Dirichlet series $D(s)$ defined in (1.3), (2) Riemann hypothesis for such $D(s)$.

For the concrete examples we take: The Dirichlet L-functions $L(s, \chi)$, the Dedekind zeta functions $\zeta_K(s)$, the Hecke L-functions, the Artin L-functions, the Selberg class, and the Elliptic L-functions $L(E, s)$ and so on in the context of the automorphic functions. And by the established analytic continuation functional equation for each function $\sum_{n=1}^{\infty} a_n/n^s$ defined in (1.3) and by the Key Lemmas hence by the formulation of (3.4) and (3.6), then for these Dirichlet series $D(s)$ we will eventually confirm the veracity of each case's: (1) Global explicit expression over $\mathbb{C}$, (2) Riemann hypothesis on the critical strip $0 < Re(s) < 1$.

Now with the natural number N given and for any integer k in $0 \leq k \leq N$, we choose a sequence of the rational numbers P_k/Q_{N_0} such that, P_0 is defined to be 0 and the other numbers P_k and Q_{N_0} are positive integers such that, for the given

$$\epsilon := \frac{1}{N^2 2^{2N+2}} \ , \tag{1.5}$$

then we have the following approximations

$$|\log(k+1) - \frac{P_k}{Q_{N_0}}| < \epsilon \ , \quad \text{for} \quad k = 0, 1, \ldots, N. \tag{1.6}$$

We let s be the variable defined on the complex plane $\mathbb{C}$, and let s_1 be the variable defined on the Riemann surface $\mathbf{S}$ of the log-function such that, on the Riemann surface $\mathbf{S}$ we define the following mapping

$$\begin{aligned} &-Q_{N_0}\log(.) : \mathbf{S} \longrightarrow \mathbb{C} \\ &-Q_{N_0}\log(.) : \tilde{s_1} \longmapsto s \\ &\quad \text{where} \ \ \tilde{s_1} : = {s_1}^{1/Q_{N_0}} := \exp(\frac{-(\sigma+it)}{Q_{N_0}}) \ , \\ &\qquad\quad s_1 : = \exp(-(\sigma+it)), \ s := (\sigma+it) \ . \end{aligned} \tag{1.7}$$

First on the Riemann surface $\mathbf{S}$ of the log-function, then by (1.7) we consider the variable $\tilde{s_1}$ $= {s_1}^{1/Q_{N_0}}$ defined on $\mathbf{S}$. Then firstly, we have the following relation

$$n^{-s} = \exp(-s\log n) = (\exp(-s))^{\log n} = (s_1)^{\log n} \ . \tag{1.8}$$

Secondly, by (1.8) we will prove that for the given positive integers n, then we will have a sequence of polynomials

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k b_{k+1} \tilde{s_1}^{P_k} \ , \tag{1.9}$$

for which, then on any compact subset contained in the region $\{s \in \mathbb{C} \mid Re(s) > 0\}$ they locally and uniformly approximate the following functions

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k b_{k+1} (k+1)^{-s} \ . \tag{1.10}$$

Now on any compact subset of the region $\{s \in \mathbb{C} \mid 0 < Re(s) < 1\}$ and by the first lemma formulated in Section 2, then for the following sequence of finite sums

$$f_N(s) := \sum_{n=0}^{N} \frac{1}{2^{n+1}} \sum_{k=0}^{n} \binom{n}{k} (-1)^k b_{k+1} (k+1)^{-s}, \tag{1.11}$$

they will locally converge uniformly to the Dirichlet η-function $f(s)$ which is defined in (3.1).

Second by (1.11), we define the finite partial sum $f_N(s_1)$ truncated from $f(s_1)$ for which it is defined on $\mathbf{S}$, namely for any given sufficiently large N, then by (1.7) we define the function $f_N(s_1)$ as

$$\begin{aligned} &f_N(s_1) : = \sum_{n=0}^{N} \frac{1}{2^{n+1}} \sum_{k=0}^{n} \binom{n}{k} (-1)^k b_{k+1} s_1^{\log(k+1)} \\ &-\log(s_1) = s = (\sigma+it) \ . \end{aligned} \tag{1.12}$$

And in considering the functional evaluation, then we have $f_N(s_1) = f_N(s)$.

We note that the complex variable s_1 is defined on $\mathbf{S}$ the Riemann surface of the log-function. Hence our change of variable pulls the domain of $f(s)$, i.e., via the $-\log(.)$-mapping, we pull the region $\{s \in \mathbb{C} \mid 0 < Re(s) \leq 1\}$ back to the Riemann sub-surface $\{s_1 \in \mathbf{S} \mid e^{-1} \leq |s_1| < 1\}$. Hence by the relation $-\log(s_1) = s$, the value of $f(s_1)$ equals the value of $f(s)$, and the value of $f_N(s_1)$ equals the value of $f_N(s)$.

And since the mapping $-\log(.)$ is a homeomorphism from $\mathbf{S}$ to $\mathbb{C}$, hence through the mapping $s_1 \mapsto -\log(s_1) = s$ it organizes a one-to-one correspondence among any compact subset given in the region $\{s_1 \in \mathbf{S} \mid e^{-1} \leq |s_1| < 1\}$ with the corresponding one given in the region $\tilde{\mathbf{C}} = \{s \in \mathbb{C} \mid 0 < Re(s) \leq 1\}$.

While by the arguments in proving the Key lemmas in Section 2, then for the partial sums $f_N(s_1) = f_N(s)$ they both approximate $f(s_1) = f(s)$ simultaneously and uniformly, on any two given compact subsets while one is given in $\{s_1 \in \mathbf{S} \mid e^{-1} \leq |s_1| < 1\}$ and the other is given in $\tilde{\mathbf{C}}$ such that, for both compact subsets they are related to each other by the corresponding mapping $s_1 \mapsto -\log(s_1) = s$.

By that we will attach to each partial sum $f_N(s_1)$, a polynomial $\tilde{f_N}(\tilde{s_1})$ for which, it will be defined on the Riemann surface $\mathbf{S}$ with the variable $\tilde{s_1} = {s_1}^{1/Q_{N_0}}$ and where for each number Q_{N_0}, it is the positive integer arranged by (1.5) and (1.6). Then we consider the polynomial $\tilde{f_N}(\tilde{s_1})$ for which, it is defined by the following

$$\begin{aligned} \tilde{f_N}(\tilde{s_1}) :&= \sum_{n=0}^{N} \frac{1}{2^{n+1}} \sum_{k=0}^{n} \binom{n}{k} (-1)^k b_{k+1} \tilde{s_1}^{P_k} \\ \tilde{s_1} &= s_1^{1/Q_{N_0}} = \exp\left(\frac{-(\sigma+it)}{Q_{N_0}}\right) , \\ -\log(s_1) &= s = (\sigma+it) . \end{aligned} \tag{1.13}$$

Then for such $\tilde{f_N}(\tilde{s_1})$, we employ it to approximate the values of two such functions $f_N(s_1) = f_N(s)$ and $f(s_1) = f(s)$. For that, $\tilde{f_N}(\tilde{s_1})$ is defined on the Riemann sub-surface $\tilde{\mathbf{S}} = \{\tilde{s_1} \in \mathbf{S} \mid e^{-1/Q_{N_0}} \leq |\tilde{s_1}| < 1\}$ for which, it is corresponding to the function $f_N(s)$ defined on the region $\tilde{\mathbf{C}} = \{s \in \mathbb{C} \mid 0 < Re(s) \leq 1\}$ through the mapping $-Q_{N_0}\log(\tilde{s_1}) \mapsto s$. In Section 4 Lemma 4.3 we will show that for such approximations, they are all locally uniform approximations.

Third on the Riemann surface $\mathbf{S}$, we have the Riemann sub-surface $\{s_1 \in \mathbf{S} \mid e^{-1} \leq |s_1| < 1\}$ for which, it is the inverse image of the region $\tilde{\mathbf{C}} = \{s \in \mathbb{C} \mid 0 < Re(s) \leq 1\}$ pulled back by the mapping $s_1 \mapsto -\log(s_1) = s$. We will restrict the defining domain of the mapping $s_1 \mapsto -\log(s_1)$ to the radial line segment $\mathbf{L} = \{s_1 \in \mathbf{S} \mid e^{-1} \leq |s_1| < 1\ ,\ \arg(s_1) = -t_0\}$ defined in the Riemann surface $\mathbf{S}$.

Now on the following Riemann sub-surfaces

$$\begin{aligned} \tilde{\mathbf{S}} :&= \{\tilde{s_1} \in \mathbf{S} \mid e^{-1/Q_{N_0}} \leq |\tilde{s_1}| < 1\} \\ \tilde{\mathbf{C}} :&= \{s \in \mathbb{C} \mid 0 < Re(s) \leq 1, Im(s) = t_0\} , \end{aligned} \tag{1.14}$$

we consider the following functional relations

$$\begin{aligned} \tilde{f_N}(\tilde{s_1}) &= \sum_{n=0}^{N} \frac{1}{2^{n+1}} \sum_{k=0}^{n} \binom{n}{k} (-1)^k b_{k+1} \tilde{s_1}^{P_k} \\ -Q_{N_0}\log(\tilde{s_1}) &= -\log(s_1) = s . \end{aligned} \tag{1.15}$$

And with respect to Lemma 4.3 and Lemma 4.5, we specify their associated given compact subsets K_s and K_c by the following relation

$$\begin{aligned} -Q_{N_0}\log(.) : K_s \longrightarrow K_c :&= \{s \in \tilde{\mathbf{C}} \mid \alpha \leq Re(s) \leq \beta,\ Im(s) = t_0\} \\ -Q_{N_0}\log(\tilde{s_1}) &= s , \end{aligned} \tag{1.16}$$

with the condition $0 < \alpha \le \beta < 1$ given and fixed. And we define the following notation

$$\lim_{(N)} := \lim_{\substack{|\log(k+1) - \frac{P_k}{Q_{N_0}}| < \frac{1}{N^2 2^{2N+2}} \\ 0 \le k \le N \\ N \to \infty}} . \tag{1.17}$$

Then on the compact subset K_c we first show that for each given positive integer m_0 and for each integer r in $0 \le r \le m_0$, we have the following locally uniform convergence

$$\begin{aligned} f^{(r)}(s) &= \lim_{(N)} \sum_{k=0}^{N} (-1)^k b_{k+1} \sum_{l=k}^{N} \binom{l}{k} \frac{1}{2^{l+1}} \frac{d^r}{ds^r} \tilde{s_1}^{P_k} \\ &\quad \text{on the given compact subset } K_c \text{ defined in (1.16)} \\ &= \lim_{(N)} \sum_{k=0}^{N} (-1)^k b_{k+1} \sum_{l=k}^{N} \binom{l}{k} \frac{1}{2^{l+1}} (-1)^r (\frac{P_k}{Q_{N_0}})^r \tilde{s_1}^{P_k} \\ &\quad \text{since } \frac{d^r}{ds^r} \tilde{s_1}^{P_k} = \frac{(-1)^r P_k(P_k-1)\dots(P_k-r+1)}{Q_{N_0}^r} \tilde{s_1}^{P_k - r}. \end{aligned} \tag{1.18}$$

Then similarly to the η-function $f(s)$, thus for each function $f^{(r)}(s)$ first we have all the similar formulations defined from (1.5) to (1.16), and then we have the following locally uniform approximations

Theorem 1.1. *On the given compact subset K_s defined in (1.16) and for all sufficiently large integers $M \ge \max\{M(K_s, m_0), N\}$, where the given and fixed integer $M(K_s, m_0)$ depends on K_s and on the functions $f^{(r)}(s)$ with all the integers r in $0 \le r \le m_0$, and where $f(s)$ is the Dirichlet η-function defined in (3.1) with each derivative $f^{(r)}(s)$, and with assuming the same conditions of Lemma 4.3. Then for each M and for each number $\vartheta(N)$*

$$\vartheta(N) := \frac{1}{2^{N/2}} + \frac{1/2}{3^{(\log(N+1))^{\lambda_0}}},$$

where λ_0 is a given fixed positive integer, we have the locally uniform estimations

(1) *The polynomials $\widetilde{f_M^{(r)}}(\tilde{s_1})$ approximate the series $f^{(r)}(\sigma + it_0)$ within a perturbation of at most $\vartheta(N)$, and we denote them by*

$$\widetilde{f_M^{(r)}}(\tilde{s_1}) \asymp f^{(r)}(\sigma + it_0)\ ,\ \tilde{s_1} = s_\sigma = \exp(\frac{-(\sigma+it)}{Q_{M_0}}) \quad \textit{within } \vartheta(N)\ . \tag{1.19}$$

(2) *For the difference polynomials $f_M^{(r)}(s_{\alpha,\beta})$ formulated similarly as those of (4.12), where $s_\sigma = \exp(\frac{-(\sigma+it)}{Q_{M_0}})$ is defined on K_s. Then they approximate the series $f^{(r)}(\alpha + it_0) - f^{(r)}(\beta + it_0)$ within a perturbation of at most $2\vartheta(N)$, and we denote them by*

$$f_M^{(r)}(s_{\alpha,\beta}) \asymp f^{(r)}(\alpha + it_0) - f^{(r)}(\beta + it_0) \quad \textit{within } 2\vartheta(N)\ . \tag{1.20}$$

Fourth for the functions $\phi(s)$ which are defined in (6.1) then by specific property of functions $\phi(s)$, we have the similar asymptotics Theorem 6.5 as those of Theorem 4.7 to obtain the result of Theorem 6.8. And with the locally uniform estimation Lemma 5.10, finally in Theorem 5.9 and Theorem 6.9 we employ Theorem 5.8 and Theorem 6.8 to solve problems of Riemann hypothesis for the Dedekind zeta functions, the Hecke L-functions, the Artin L-functions, and the Selberg class for which all of their nontrivial zeros are contained in the vertical line $Re(s) = 1/2$.

Finaly for the $\gamma(s)$-factor of each Dirichlet series $D(s)$ which is formulated in (1.4), then by Theorem 6.9 it has neither zeros nor poles contained in the critical strip $\{s \in \mathbb{C} \mid 0 < Re(s) < 1\}$ and the non-existence of Siegel's zeros for such Dirichlet series $D(s)$ is confirmed.

2. Key Lemmas

For our purpose to explore the infinite series which is expressed on the right hand side of (1.4), we begin with studying the following difference formulas. We take the given complex numbers b_n defined in (1.4), and define

$$\Delta_n(\frac{b_\nu}{\nu^s}) := \sum_{k=0}^{n} \binom{n}{k} (-1)^k \frac{b_{k+1}}{(k+1)^s} \ . \tag{2.1}$$

Our method of estimating (2.1) is under the given fixed real number t in the complex variable $s = \sigma + it$, by which we pull the original estimation in the complex plane $\mathbb{C}$ back to the Riemann surface $\mathbf{S}$, for the chosen natural number Q_{N_0} with $N \geq n$, through the mapping

$$\begin{aligned} &-Q_{N_0} \log(.) : \mathbf{S} \longrightarrow \mathbb{C} \\ &-Q_{N_0} \log(.) : \tilde{s_1} \longmapsto s \\ &\text{where } \ \tilde{s_1} : = {s_1}^{1/Q_{N_0}} := \exp(\frac{-(\sigma+it)}{Q_{N_0}}) \ , \\ &s_1 : = \exp(-(\sigma+it)), \ s := (\sigma+it) \ . \end{aligned} \tag{2.2}$$

Hence for the complex variable $s_\sigma := \tilde{s_1} = \exp(\frac{-(\sigma+it)}{Q_{N_0}})$ defined in (2.2), it is the chosen variable on the Riemann surface $\mathbf{S}$, whose corresponding variable

$$w_\sigma := -Q_{N_0} \log(s_\sigma) = -Q_{N_0} \log(\tilde{s_1}) = \sigma + it \ , \tag{2.3}$$

is defined on the complex plane $\mathbb{C}$.

On the other hand, for each natural number n with $n \leq N$ and for any integer k in $0 \leq k \leq n$, we choose a sequence of the rational numbers P_k/Q_{n_0} such that, for the integers P_k and Q_{n_0}, and for the given

$$\epsilon := \frac{1}{n^2 2^{2n+2}} \ , \tag{2.4}$$

then we have the following approximations

$$|\log(k+1) - \frac{P_k}{Q_{n_0}}| < \epsilon \ , \quad \text{for} \ \ 0 \leq k \leq n \ . \tag{2.5}$$

Furthermore for any two complex points $\alpha + it_0$ and $\beta + it_0$ contained in the given horizontal line $t = t_0$, while by substituting n for N in formulations (2.2) and (2.3), then we have the similar expression of (2.3)

$$- Q_{n_0} \log(s_\alpha) = \alpha + it_0 \quad \text{and} \quad - Q_{n_0} \log(s_\beta) = \beta + it_0 \ , \tag{2.6}$$

where s_α and s_β are complex numbers defined in the Riemann surface $\mathbf{S}$. And by the setting of (2.2) restricted to the Riemann sub-surface $\tilde{\mathbf{S}} = \{\tilde{s_1} \in \mathbf{S} \mid e^{-1/Q_{n_0}} \leq |\tilde{s_1}| < 1\}$, then by (2.1) with (2.5) and (2.6), we define the following polynomial

$$\begin{aligned} \Delta_n(\tilde{s_1}) : = \sum_{k=0}^{n} \binom{n}{k} (-1)^k b_{k+1} {\tilde{s_1}}^{P_k} \\ \tilde{s_1} = s_\alpha \text{ or } s_\beta \ . \end{aligned} \tag{2.7}$$

Now by (2.6) and (2.7) we consider the difference polynomial

$$\Delta_n(s_\alpha) - \Delta_n(s_\beta) \ , \tag{2.8}$$

for which, then since the following relation

$$s_\alpha^{P_k} = \sum_{j=1}^{P_k} \binom{P_k}{j} (s_\alpha s_\beta^{-1} - 1)^j s_\beta^{P_k} + s_\beta^{P_k} \ , \tag{2.9}$$

by which the difference polynomial is precisely expressed as

$$\begin{aligned} &\Delta_n(s_\alpha) - \Delta_n(s_\beta) \\ &= \sum_{j=1}^{P_n} (\beta-\alpha)^j \exp(\frac{-j\sigma_{\alpha,\beta}}{Q_{n_0}}) \sum_{k=0}^{n} \frac{1}{(Q_{n_0})^j} \binom{P_k}{j} (-1)^k b_{k+1} \\ &\qquad \exp(\frac{-(\beta+it_0)P_k}{Q_{n_0}}) \sum_{l=k}^{n} \binom{l}{k} \frac{1}{2^{l+1}} \ , \\ &\binom{P_k}{j} := 0 \quad \text{if } j > P_k \ , \end{aligned} \tag{2.10}$$

where the positive real numbers $\sigma_{\alpha,\beta}$ with $\alpha < \sigma_{\alpha,\beta} < \beta$, is derived from the Mean Valued Theorem applying to

$$\begin{aligned} \exp(\frac{-\alpha}{Q_{n_0}}) - \exp(\frac{-\beta}{Q_{n_0}}) &= (\exp(\frac{(\beta-\alpha)}{Q_{n_0}}) - 1) \exp(\frac{-\beta}{Q_{n_0}}) \\ &= (\frac{\beta-\alpha}{Q_{n_0}}) \exp(\frac{-\sigma_{\alpha,\beta}}{Q_{n_0}}) \ . \end{aligned} \tag{2.11}$$

Furthermore for any given k with $0 \le k \le n$, so long as all the rational numbers $\frac{P_k}{Q_{n_0}} \to \log(k+1)$ simultaneously, then it implies that

$$\begin{aligned} (\frac{1}{Q_{n_0}})^j \binom{P_k}{j} &= \frac{(P_k)(P_k-1)\dots(P_k-j+1)}{(Q_{n_0})^j (j!)} \\ &\longrightarrow \frac{(\log(k+1))^j}{j!} \ , \end{aligned} \tag{2.12}$$

is strictly increasing in a sequence of positive real numbers.

Meanwhile by applying (2.12) to (2.10), then we have the following asymptotic formula

$$\begin{aligned} &\Delta_n(s_\alpha) - \Delta_n(s_\beta) \\ &\asymp \sum_{j=1}^{\infty} (\beta-\alpha)^j \sum_{k=0}^{n} \frac{(\log(k+1))^j}{j!} (-1)^k b_{k+1} \\ &\qquad \exp(-(\beta+it_0)\log(k+1)) \sum_{l=k}^{n} \binom{l}{k} \frac{1}{2^{l+1}} \end{aligned} \tag{2.13}$$

which will take care of the estimation of the difference polynomial $\Delta_n(s_\alpha) - \Delta_n(s_\beta)$ defined on the Riemann sub-surface $\tilde{\mathbf{S}} = \{\tilde{s_1} \in \mathbf{S} \mid e^{-1/Q_{n_0}} \le |\tilde{s_1}| < 1\}$.

We note that by the arguments of (2.12) and (2.13), the right hand side polynomial is a dominant asymptotic w.r.t. the left hand side difference polynomial $\Delta_n(s_\alpha) - \Delta_n(s_\beta)$. Namely the domination is decided by comparing all the absolute values of each pair of the corresponding coefficients interpreted in (2.13) w.r.t. the same monomial term $(\beta-\alpha)^j$.

After the above construction, we are back to estimate the differernces defined in (2.1). Firstly for the given natural numbers m and n, and for the natural number n which is sufficiently large as required in (2.16) below, then we have the approximations defined in (2.5). Secondly by the pull-back relation $w_\alpha = -Q_{n_0} \log(s_\alpha)$ defined in (2.3), and by the pull-back mapping defined in

(2.2), then for each given w_α in the region $\tilde{\mathbf{C}} = \{s \in \mathbb{C} \mid 0 < Re(s) \leq 1\}$ and for its corresponding s_α in the Riemann sub-surface $\tilde{\mathbf{S}}$ we define

$$\begin{aligned} \Delta_n(w_\alpha) &:= \Delta_n(\frac{b_\nu}{\nu^{w_\alpha}}) = \sum_{k=0}^{n} \binom{n}{k} (-1)^k \frac{b_{k+1}}{(k+1)^{w_\alpha}} \quad \text{on } \tilde{\mathbf{C}} \ , \\ \Delta_n(s_\alpha) &:= \sum_{k=0}^{n} \binom{n}{k} (-1)^k b_{k+1} s_\alpha^{P_k} \quad \text{on } \tilde{\mathbf{S}} = \{s_\alpha \in \mathbf{S} \mid e^{-1/Q_{n_0}} \leq |s_\alpha| < 1\}\} \ . \end{aligned} \tag{2.14}$$

Then we have the following four fundamental lemmas.

Lemma 2.1 (Key lemma (1)). *For each given complex point $w_\alpha = \alpha + it_0$, and for each given compact subset K_c which contains w_α is also contained in the region $\{w_\sigma \in \tilde{\mathbf{C}} \mid Im(w_\sigma) = t_0\}$. We suppose K is the number*

$$K := \max_{w_\sigma \in K_c} \{|w_\sigma|\} \ , \tag{2.15}$$

and we suppose for each integer n such that n satisfies

$$n \geq K + 1 \ . \tag{2.16}$$

Then for each given real number $p > 1 + |\alpha|$ there always exists a real number

$$n_n := \max\{4(\log(n+1))^2 \ , \ e(p-\alpha)^2\} \ , \tag{2.17}$$

such that, by this real number n_n we always have the following estimation

$$\begin{aligned} n^{-\varepsilon}|\Delta_n(w_\alpha)| < &\frac{1}{2^n} \\ &+ \ n_n(p-\alpha)^{n_n}(\log(n_n+1))^{n_n} \ + \ n \\ &+ \ n^{-\varepsilon}|\Delta_n(w_p)| \ , \end{aligned} \tag{2.18}$$

where $e = 2.71828\ldots$ is the Euler number, $w_p = p + it_0$ and $b_n = o(n^\varepsilon)$ for any $\varepsilon > 0$.

Proof. By applying the triangular inequality to $|\Delta_n(w_\alpha)|$ then we have

$$\begin{aligned} n^{-\varepsilon}|\Delta_n(w_\alpha)| \ \leq &\ n^{-\varepsilon}|\Delta_n(w_\alpha) - \Delta_n(s_\alpha)| \\ &+ \ n^{-\varepsilon}|\Delta_n(s_\alpha) - \Delta_n(s_p)| \ + \ n^{-\varepsilon}|\Delta_n(s_p) - \Delta_n(w_p)| \\ &+ \ n^{-\varepsilon}|\Delta_n(w_p)| \ . \end{aligned} \tag{2.19}$$

And we are going to estimate the right hand side of (2.19).

By definition (2.14) the first term is

$$| \sum_{k=0}^{n} \binom{n}{k} (-1)^k b_{k+1} \{\exp(-w_\alpha \log(k+1)) - \exp(\frac{-w_\alpha P_k}{Q_{n_0}})\} | \ ,$$

among which, the absolute value of each factor

$$\exp(-w_\alpha \log(k+1)) - \exp(\frac{-w_\alpha P_k}{Q_{n_0}}) \ ,$$

for which, it is due to applying the Taylor expansion to the exponential function $\exp(-w_\alpha \log(k+1))$ w.r.t. the point $-w_\alpha P_k/Q_{n_0}$, then it is less than $\frac{1}{2^{2n+1}}$, since in (2.16) we have assumed $n \geq K+1$ and in (2.5) the $(n+1)$-many approximations

$$|\log(k+1) - \frac{P_k}{Q_{n_0}}| < \frac{1}{n^2 2^{2n+2}} \ , \quad 0 \leq k \leq n \ .$$

Therefore by $a_n = o(n^{\varepsilon})$ we have the estimation

$$n^{-\varepsilon}|\Delta_n(w_\alpha) - \Delta_n(s_\alpha)| < \frac{1}{2^{n+1}} \ . \tag{2.20}$$

For the second term, by the difference polynomial $\Delta_n(s_\alpha) - \Delta_n(s_p)$ interpreted in (2.10) and by the argument in (2.13), then $n^{-\varepsilon}\ |\Delta_n(s_\alpha) - \Delta_n(s_p)|$ is dominated by the following

$$\begin{aligned}&\sum_{j=1}^{P_n} |(p-\alpha)^j \exp(\frac{-j\sigma_{\alpha,p}}{Q_{n_0}})| \\ &\qquad \sum_{k=0}^{n} |\frac{(\log(k+1))^j}{j!}(-1)^j \exp(\frac{-(p+it_0)P_k}{Q_{n_0}})| \ ,\end{aligned} \tag{2.21}$$

where the real number $\sigma_{\alpha,p}$ is derived from the Mean Valued Theorem by applying it to the difference

$$\exp(-\frac{\alpha}{Q_{n_0}}) - \exp(-\frac{p}{Q_{n_0}}) = (\frac{p}{Q_{n_0}} - \frac{\alpha}{Q_{n_0}})\exp(\frac{-\sigma_{\alpha,p}}{Q_{n_0}}) \ .$$

Moreover for the estimation (2.21) since $p > 1 + |\alpha|$ is chosen, so it is dominated by the following quantity

$$\begin{aligned}&n_n(p-\alpha)^{n_n}(\log(n_n+1))^{n_n} + \sum_{j=[n_n]+1}^{P_n}\sum_{k=0}^{n}(p-\alpha)^j\frac{(\log(k+1))^j}{j!} \\ &= n_n(p-\alpha)^{n_n}(\log(n_n+1))^{n_n} + \sum_{j=[n_n]+1}^{P_n}\frac{(p-\alpha)^j}{\sqrt{j!}}\sum_{k=0}^{n}\frac{(\log(k+1))^j}{\sqrt{j!}} \ ,\end{aligned}$$

where the number n_n will be decided in the followings.

Firstly the number n_n will be decided by the following three observations:

- (1) if $j > e(p-\alpha)^2$, then $\frac{(p-\alpha)^j}{\sqrt{j!}} < 1$ where e is the Euler number
- (2) if $j > e(\log(k+1))^2$, then $\frac{(\log(k+1))^j}{\sqrt{j!}} < 1$
- (3) if $\sqrt{j} > 2\log(k+1)$, then $\frac{\log(k+1)}{\sqrt{j}} < \frac{1}{2}$

while these three are due to applying Stirling's formula:

$$(j-1)! \asymp \sqrt{\frac{2\pi}{j}}e^{j(\log j - 1)}(1 + \frac{1}{12j} + O(\frac{1}{j^2})) \ .$$

Namely, for each given sufficiently large integer j, if $j > e(p-\alpha)^2$ then

$$\frac{1}{2}\log(j!) > \frac{1}{2}(j\log(j) - j) = \frac{j}{2}\log(\frac{j}{e}) > j\log(p-\alpha) \ ,$$

which implies (1). Similarly for each sufficiently large $j > e(\log(k+1))^2$ then

$$\frac{1}{2}\log(j!) > \frac{1}{2}(j\log(j) - j) > j\log(\log(k+1)) \ ,$$

which implies (2).

Therefore by these estimations, it implies that if we take the following number

$$n_n := \max\{4(\log(n+1))^2 \ , \ e(p-\alpha)^2\} \ ,$$

then by the estimation of formula (2.21), the second term of (2.19) is dominated by the dominating number

$$
\begin{aligned}
& n_n(p-\alpha)^{n_n}(\log(n_n+1))^{n_n} \\
& \qquad + \sum_{j=[n_n]+1}^{P_{n+1}} \frac{(p-\alpha)^j}{\sqrt{j!}} \sum_{k=0}^{n} \frac{(\log(k+1))^j}{\sqrt{j!}} \\
& < n_n(p-\alpha)^{n_n}(\log(n_n+1))^{n_n} + n \sum_{\ell=1}^{P_{n+1}-[n_n]} (\frac{1}{2})^{\ell} \\
& < n_n(p-\alpha)^{n_n}(\log(n_n+1))^{n_n} + n \ .
\end{aligned}
\tag{2.22}
$$

Meanwhile, the estimation of the third term is similar to the estimation of the first term, since we carry out it by substituting p for α in the argument by which we have estimated the first term of (2.19) in the above.

Therefore putting (2.20) (2.21) (2.22) together, then we have the following inequality

$$
\begin{aligned}
n^{-\varepsilon}|\Delta_n(w_\alpha)| < & \frac{1}{2^{n+1}} \\
& + n_n(p-\alpha)^{n_n}(\log(n_n+1))^{n_n} + n \\
& + \frac{1}{2^{n+1}} \\
& + n^{-\varepsilon}|\Delta_n(w_p)| \ .
\end{aligned}
$$

Therefore we have proved the estimation of (2.18). □

Lemma 2.2 (Key lemma (2))**.** *For each point w_α contained in the given compact subset $K_c \subset \{w_\sigma \in \tilde{\mathbf{C}} \mid Im(w_\sigma) = t_0\}$, and we assume all the conditions of Lemma 2.1. Then there exists a fixed minimum integer $n_0(N)$ for which it depends on K_c and on the infinite sequence of positive real numbers $\{p = p_n \mid p_n = 3(\log(n+1))^{\lambda_0} + K,\ \lambda_0 :=$ a given fixed positive integer $\geq 2\}$, such that $n_0(N) \geq N \geq K+1$ where N is a given fixed integer and*

$$
\begin{aligned}
|\sum_{n\geq n_0(N)}^{\infty} \frac{1}{2^{n+1}}\Delta_n(w_\alpha)| < & \frac{1}{2^{\frac{N}{2}+1}} + |\sum_{n\geq n_0(N)}^{\infty} \frac{1}{2^{n+1}}\Delta_n(w_p)| \\
& < \frac{1}{2^{N/2}} + \frac{1/2}{3^{(\log(N+1))^{\lambda_0}}} \ ,
\end{aligned}
\tag{2.23}
$$

where each $\Delta_n(w_\alpha)$ is defined in (2.14).

Proof. By (2.15) and (2.16) we choose the integer $n_0(N)$ to be the minimum in the infinite sequence $\{n\}$ of positive integers such that, for all integers n in $n \geq n_0(N) \geq N \geq K+1$ together by (2.18) they satisfy the following estimations:

$$
\begin{aligned}
& \sum_{n\geq n_0(N)}^{\infty} \frac{1}{2^{n+1}} n^{\varepsilon}\{\frac{1}{2^n} + n_n(p-\alpha)^{n_n}(\log(n_n+1))^{n_n} + n\} < \frac{1}{2^{1+N/2}} \ , \\
& \sum_{n\geq n_0(N)}^{\infty} \frac{1}{2^{n+1}}|\Delta_n(w_p)| < \frac{1}{2^{1+N/2}} + \frac{1/2}{3^{(\log(N+1))^{\lambda_0}}} \ , \text{ for all integers } n \geq n_0(N) \ ,
\end{aligned}
\tag{2.24}
$$

for which by (2.18) and for each n with $|b_{n+1}(n+1)^{-\varepsilon}| < 1$, we let $p = p_n = 3(\log(n+1))^{\lambda_0} + K$, $\lambda_0 :=$ a given fixed positive integer ≥ 2 then by (2.17)

$$\begin{aligned} & 0 < \frac{1}{2^{\frac{n}{4}}}\{n(p-\alpha)^{(\log(n+1))^{\lambda_0}}(\lambda_0\log(n+1))^{(\log(n+1))^{\lambda_0}}\} \to 0\ ,\ \text{as } n\to\infty, \\ & \frac{n^{\varepsilon}}{2^{\frac{n}{4}}}\{n(p-\alpha)^{4(\log(n+1))^{2\lambda_0}}(2\lambda_0\log(n+1))^{4(\log(n+1))^{2\lambda_0}}\} < 1,\ \text{for all integers } n \geq n_0(N). \end{aligned} \tag{2.25}$$

This implies the first estimation of (2.24).

While for the second estimation of (2.24) and for each n with $|b_{j+1}(n+1)^{-\varepsilon}| < 1$ for each j in $0 \leq j \leq n$, we let $p = p_n = 3(\log(n+1))^{\lambda_0} + K$, $\lambda_0 :=$ a given fixed positive integer ≥ 2 then

$$n^{-\varepsilon}|\Delta_n(w_p)| < 1 + \frac{n}{2^{p_n}} + n2^n\frac{1}{3^{p_n}},\ \text{since } \binom{n}{j} < 2^n,$$

$$\begin{aligned} \sum_{n\geq n_0(N)}^{\infty}\frac{1}{2^{n+1}}|\Delta_n(w_p)| &< \sum_{n\geq n_0(N)}^{\infty}\frac{n^{\varepsilon}}{2^{n+1}}(1+n) + \frac{1}{2}\sum_{n\geq n_0(N)}^{\infty}\frac{n^{1+\varepsilon}}{3^{p_n}} \\ &< \frac{1}{2^{1+N/2}} + \frac{1/2}{3^{(\log(n_0(N)+1))^{\lambda_0}}}\sum_{n\geq n_0(N)}^{\infty}\frac{n^{1+\varepsilon}}{3^{2(\log(n+1))^{\lambda_0}}}, \end{aligned} \tag{2.26}$$

$$\sum_{n\geq n_0(N)}^{\infty}\frac{n^{1+\varepsilon}}{3^{2(\log(n+1))^{\lambda_0}}} < 1\ ,\ \text{for all integers } n \geq n_0(N) \text{ since } \lambda_0 \geq 2.$$

This implies the second estimation of (2.24) since the condition $n_0(N) \geq N$.

Now we claim that: The estimations of (2.24) is based on estimations of (2.25) and (2.26), by which we have the existence of integer $n_0(N)$ with which it will satisfy estimation (3.23).

First, we base on $\Delta_n(w_\alpha)$ which is equal to

$$(\Delta_n(w_\alpha) - \Delta_n(s_\alpha)) + (\Delta_n(s_\alpha) - \Delta_n(s_p)) + (\Delta_n(s_p) - \Delta_n(w_p)) + \Delta_n(w_p)$$

with which, we apply it to estimating the following infinite summation

$$|\sum_{n\geq n_0(N)}^{\infty}\frac{1}{2^{n+1}}\Delta_n(w_\alpha)|\ ,$$

for which, then by the triangular inequality, it is less than

$$\begin{aligned} |\sum_{n\geq n_0(N)}^{\infty}\frac{1}{2^{n+1}}\{(\Delta_n(w_\alpha)-\Delta_n(s_\alpha))\ +\ (\Delta_n(s_\alpha)-\Delta_n(s_p)) \\ +\ (\Delta_n(s_p)-\Delta_n(w_p))\}|\ +\ |\sum_{n\geq n_0(N)}^{\infty}\frac{1}{2^{n+1}}\Delta_n(w_p)|\ , \end{aligned}$$

for which, then by (2.18) it is less than

$$\sum_{n\geq n_0(N)}^{\infty}\frac{n^{\varepsilon}}{2^{n+1}}\{\frac{1}{2^n} + n_n(p-\alpha)^{n_n}(\log(n+1))^{n_n} + n\}\ +\ \sum_{n\geq n_0(N)}^{\infty}\frac{1}{2^{n+1}}|\Delta_n(w_p)|\ . \tag{2.27}$$

Therefore by applying estimations of (2.25) and (2.26) to the estimation of (2.27) with the minimum integer $n_0(N) \leq n$ conditioned, with $|b_{j+1}(n+1)^{-\varepsilon}| < 1$ for each j in $0 \leq j \leq n$ and with $p = p_n = 3(\log(n+1))^{\lambda_0} + K$ while $n_0(N) \geq N \geq K+1$ w.r.t (2.16), so it implies

$$\sum_{n \geq n_0(N)}^{\infty} \frac{n^{\varepsilon}}{2^{n+1}} \{ \frac{1}{2^n} + n_n (p-\alpha)^{n_n} (\log(n+1))^{n_n} + n + n^{-\varepsilon} |\Delta_n(w_p)| \} < \frac{2}{2^{1+N/2}} + \frac{1/2}{3^{(\log(N+1))^{\lambda_0}}} \ , \tag{2.28}$$

hence we complete the proof of estimation (2.23). □

We note that: For each given given N, with which we define the minimum $n_0(N)$ in (2.24) which will eventually be less than N if N is large enough, since by substituting N for $n_0(N)$ in the infinite sum which is on the left hand side of the inequality (2.28), then we first combine the inequality of (2.25) with (2.24), and then combine the inequalities of (2.26) with (2.24), hence together it implies $N \geq n_0(N)$ eventually. Hence eventually we may identify N with $n_0(N)$.

Lemma 2.3 (Key lemma (3))**.** *For any given $w_\alpha = \alpha + it_0$ and $w_\beta = \beta + it_0$ contained in the given compact subset $K_c \subset \{w_\sigma \in \tilde{\mathbf{C}} \mid Im(w_\sigma) = t_0\}$, and we assume all the conditions of Lemma 2.1. Then there exists a fixed minimum integer $n_0(N)$ for which it depends on K_c and on the infinite sequence of positive real numbers $\{p = p_n \mid p_n = 3(\log(n+1))^{\lambda_0} + K,\ \lambda_0 :=$ a given fixed positive integer $\geq 2\}$ where $K = \max_{w_\sigma \in K_c}\{|w_\sigma|\}$, such that $n_0(N) \geq N \geq K+1$ where N is a given fixed integer and*

$$| \sum_{n \geq n_0(N)}^{\infty} \frac{1}{2^{n+1}} \Delta_n(w_\beta) - \sum_{n \geq n_0(N)}^{\infty} \frac{1}{2^{n+1}} |\Delta_n(w_\alpha)| < \frac{2}{2^{N/2}} + \frac{1}{3^{(\log(N+1))^{\lambda_0}}} \ , \tag{2.29}$$

where each $\Delta_n(w_\alpha)$ is defined in (2.14).

Proof. By applying Lemma 2.2 twice, then we have the proof. □

Lemma 2.4 (Key lemma (4))**.** *For any given four points $w_\alpha = \alpha + it_0$, $w'_\beta = \beta + it'_0$, $w'_p := p + it'_0$, and $w_p := p + it_0$ which are contained in the right half complex plane $Re(s) > 0$, and we assume all the conditions of Lemma 2.1. Then there exists a fixed minimum integer $n_0(N)$ for which it depends on $K_c := \{w_\alpha, w'_\beta, w'_p, w_p\}$ and on the infinite sequence of positive real numbers $\{p = p_n \mid p_n = 3(\log(n+1))^{\lambda_0} + K_1, \lambda_0 :=$ a given fixed positive integer $\geq 2\}$ where $K_1 := \max\{|w_\alpha|, |w'_\beta|, |w'_p|, |w_p|\}$, such that $n_0(N) \geq N \geq K_1+1$ where N is a given fixed integer and*

$$| \sum_{n \geq n_0(N)}^{\infty} \frac{1}{2^{n+1}} \Delta_n(w'_\beta) - \sum_{n \geq n_0(N)}^{\infty} \frac{1}{2^{n+1}} |\Delta_n(w_\alpha)| < \frac{2}{2^{N/2}} + \frac{1}{3^{(\log(N+1))^{\lambda_0}}} \ , \tag{2.30}$$

where each $\Delta_n(w_\alpha)$ is defined in (2.14).

Proof. By applying Lemma 2.2 twice, then we have the proof. □

3. Hasse-Weil Conjecture

For the concrete examples of the automorphic functions we likely encounter: the Riemann zeta function $\zeta(s)$, the Dirichlet L-functions $L(s, \chi)$, the Dedekind zeta functions $\zeta_K(s)$, the Hecke L-functions, the Artin L-functions, the Selberg class and so on in the context of the automorphic functions. Then for the most prominent property of them by which we define

Definition 3.1. For those Dirichlet series $D(s)$ defined in (1.3), we specify a class $\mathbf{D}$ of members recruited from them, by further assuming that: for each $D(s) \in \mathbf{D}$, there is a non-negative integer d_0 such that the function

$$f(s) := (1 - 2^{1-s})^{d_0} D(s) \ , \ Re(s) > 1 \ , \tag{3.1}$$

has the unique analytic continuation $f(s)$ which is an entire function defined over the complex plane $\mathbb{C}$. And we call such $f(s)$ the η-function of $D(s)$. We also assume that for each $D(s)$ it has its own functional equation with its own $\gamma(s)$- factor in its analytic continuation to the whole complex plane.

In the following we will show that for the η-function $f(s)$ of each $D(s) \in \mathbf{D}$ then by the Key lemmas of Section 2, $f(s)$ has an explicit infinite sum such that it is an analogue to the formulations of (1.1) and (1.2). Firstly we rearrange the infinite sum of (3.1) as

$$f(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} b_n}{n^s} \ , \ Re(s) > 1 \ , \tag{3.2}$$

which is also a member of the class $\mathbf{D}$. Secondly by applying the Euler transformation to the infinite sum of (3.2), then we have the infinite sum

$$f(s) = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \sum_{k=0}^{n} \binom{n}{k} (-1)^k b_k (k+1)^{-s} \ , \ Re(s) > 1 \ , \tag{3.3}$$

which, by the Key lemmas of Section 2, converges absolutely and uniformly on any compact subset contained in the right half complex plane $Re(s) > 0$, hence the infinite series on the right hand side of (3.3) is an analytic function defined on the right half complex plane $Re(s) > 0$. Thus by Principle of analytic continuation applying here then for the function $f(s)$, for which it is defined in (3.1), it has the following explicit analytic continuation

$$f(s) = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \sum_{k=0}^{n} \binom{n}{k} (-1)^k b_{k+1} (k+1)^{-s} \ , \ Re(s) > 0 \ . \tag{3.4}$$

Moreover for each positive integer r, since we have the well-defined Dirichlet η-function

$$f^{(r)}(s) = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \sum_{k=0}^{n} \binom{n}{k} (-1)^k b_{k+1} (-1)^r (\log(k+1))^r (k+1)^{-s} \ , \ Re(s) > 1 \ , \tag{3.5}$$

and since $b_{k+1}(-1)^r(\log(k+1))^r = o((k+1)^{\varepsilon})$ for any $\varepsilon > 0$, hence by the Key lemmas of Section 2 we also have the explicit expression

$$f^{(r)}(s) = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \sum_{k=0}^{n} \binom{n}{k} (-1)^k b_{k+1} (-1)^r (\log(k+1))^r (k+1)^{-s} \ , \ Re(s) > 0 \ , \tag{3.6}$$

which converges absolutely and uniformly on any compact subset contained in the right half complex plane $Re(s) > 0$.

4. Locally Uniform Approximations

We begin by changing the order of the double summation in the formula defined in (1.13) namely,

$$\begin{aligned} \tilde{f_N}(\tilde{s_1}) &= \sum_{n=0}^{N} \frac{1}{2^{n+1}} \sum_{k=0}^{N} \binom{n}{k} (-1)^k b_{k+1} \tilde{s_1}^{P_k} \\ &= \sum_{k=0}^{N} \sum_{l=k}^{N} \binom{l}{k} (-1)^k b_{k+1} \frac{1}{2^{l+1}} \tilde{s_1}^{P_k} \ . \end{aligned} \tag{4.1}$$

Then we have

Lemma 4.1. *For each natural number N, and for the sequence $\{a_0, a_1, \ldots, a_N\}$*

$$a_k := |\sum_{l=k}^{N} \binom{l}{k} (-1)^k b_{k+1} \frac{1}{2^{l+1}}| \ , \tag{4.2}$$

then it is a sequence of non-negative real numbers dominated by the number $N^{1+\varepsilon}$.

Proof. Since $2^{l+1} > 2^l = \binom{l}{k} + \{(1+1)^l - \binom{l}{k}\} \geq \binom{l}{k}$, hence $0 \leq a_k < N^{1+\varepsilon}$, hence it implies the proof of this lemma. □

Now with respect to the mapping $s_1 \mapsto -\log(s_1)$ and by (2.14), we first redefine the following Riemann sub-surfaces

$$\tilde{\mathbf{S}} := \{s_\sigma \in \mathbf{S} \mid e^{-1} < |s_\sigma| < 1\} \ , \quad \tilde{\mathbf{C}} := \{s \in \mathbb{C} \mid 0 < Re(s) < 1\} \ . \tag{4.3}$$

Then by the formulations of (1.6) and (1.7) and for the given positive integer Q_{N_0} we redefine the mapping

$$\begin{aligned} &-Q_{N_0}\log(.) : \tilde{\mathbf{S}} \longrightarrow \tilde{\mathbf{C}} \\ &-Q_{N_0}\log(.) : \tilde{s_1} \longmapsto w_\sigma \ , \\ &\text{where} \ \ \tilde{s_1} := {s_1}^{1/Q_{N_0}} := \exp(\frac{-(\sigma+it)}{Q_{N_0}}) \ , \\ &s_1 := \exp(-s), \ s := (\sigma+it) \ , \\ &s_\sigma := \tilde{s_1} \ , \quad w_\sigma := s = \sigma + it \ . \end{aligned} \tag{4.4}$$

And by (1.11) and (1.12) with $N \to \infty$ we have the equality

$$f(s_1) = f(s) \ . \tag{4.5}$$

Furthermore, for the two given compact subsets K_s defined in $\tilde{\mathbf{S}}$ and K_c defined in $\tilde{\mathbf{C}}$, we relate them in the following and also define the numbers $\vartheta(N)$

Definition 4.2.

$$\begin{aligned} &-Q_{N_0}\log(.) : K_s \longrightarrow K_c \ , \quad -Q_{N_0}\log(s_\sigma) = w_\sigma = s = \sigma+it \ , \\ &K_c := \{s \in \tilde{\mathbf{C}} \mid 0 < \alpha \leq Re(s) \leq \beta < 1 \ , \ Im(s) = t_0\} \ ; \\ &\vartheta(N) := \frac{1}{2^{N/2}} + \frac{1/2}{3^{(\log(N+1))^{\lambda_0}}} \ . \end{aligned} \tag{4.6}$$

Lemma 4.3. *On any compact subset K_s of the Riemann sub-surface $\tilde{\mathbf{S}}$ defined in* (4.3) *with its $-Q_{N_0}\log$ (.) mapping image K_c falls into the region $\tilde{\mathbf{C}}$. And for the minimum integer $n_0(N)$ formulated in (2.24) where the integers N satisfy $N \geq K+1$ with $K = \sqrt{1+t_0^2}$. Then for all integers M with $M \geq \max\{n_0(N), N\}$ we have*

1. *For these polynomials $\tilde{f}_M(\tilde{s_1})$ defined in* (4.1) *which also enjoy the uniformly approximating as those of the series $f_M(s_1)$ have, and approximate the series $f(s_1)$ defined in* (4.5) *within a perturbation of at most $\vartheta(N)$ by the pull-back mapping $-Q_{M_0}\log(.)$ on the Riemann sub-surface $\tilde{\mathbf{S}}$ where the numbers $\vartheta(N)$ are defined in (4.6).*
2. *By* (4.5) *in evaluation, the value of the series $f(s_1)$ on $\tilde{\mathbf{S}}$, equals the value of the series $f(s)$ on $\tilde{\mathbf{C}}$. Then as those of the series $f_M(s)$ have enjoyed, these polynomials $\tilde{f}_M(\tilde{s_1})$ approximate the series $f(s)$, by the pull-back mapping $-Q_{M_0}\log(.)$ within a perturbation of at most $\vartheta(N)$, on the compact subset K_c contained in the region $\tilde{\mathbf{C}}$ where the numbers $\vartheta(N)$ are defined in (4.6).*

Proof. For each sufficiently large integer M with $M \geq \max\{n_0(N), N\}$, then by (4.6) the variables s and $\tilde{s_1}$ are related in

$$s \in K_c = -Q_{M_0}\log(K_s) \iff s = -Q_{M_0}\log(\tilde{s_1}) \ , \ \tilde{s_1} \in K_s \ ,$$

and for each M then the difference $f_M(s) - \tilde{f}_M(\tilde{s_1})$ is equal to

$$\sum_{k=0}^{M}(-1)^k b_{k+1}\{\exp(-s\log(k+1)) - \exp(\frac{-sP_k}{Q_{M_0}})\}\sum_{l=k}^{M}\binom{l}{k}\frac{1}{2^{l+1}} \ ,$$

among which, the absolute value of each factor

$$\exp(-s\log(k+1)) - \exp(\frac{-sP_k}{Q_{M_0}}) \ ,$$

is less than $\frac{1}{2^{2M+1}}$, for which it is due to applying the Taylor expansion to the exponential function $\exp(-s\log(k+1))$ w.r.t. the point $-sP_k/Q_{M_0}$, and by the fact that we have assumed the following $(M+1)$-many approximations

$$|\log(k+1) - \frac{P_k}{Q_{M_0}}| < \frac{1}{M^2 2^{2M+2}} \ , \quad 0 \leq k \leq M \ ,$$

while we have assumed the variable s to be in the given compact subset K_c and the given integers N with $N \geq 1 + \sqrt{1+t_0^2} > 1 + |s|$. Therefore by Lemma 4.1 and by $M \geq N$, then it implies

$$|f_M(s) - \tilde{f}_M(\tilde{s_1})| < \frac{1}{2^{M+1}} \leq \frac{1}{2^{N+1}} \ . \tag{4.7}$$

Now for each given point $w_\alpha = s \in K_c$ and for each given real number $K = \sqrt{1+t_0^2} > |s|$, we employ Lemma 2.1 and Lemma 2.2. Namely, for each given number K, and then by definition (2.14), firstly by Lemma 4.1 and for each integer $n \geq K+1$ and by (2.18) then it implies

$$\begin{aligned} n^{-\varepsilon}|\Delta_n(w_\alpha)| < \frac{1}{2^n} &+ n_n(p-\alpha)^{n_n}(\log(n_n+1))^{n_n} + n \\ &+ n^{-\varepsilon}|\Delta_n(w_p)| \ , \end{aligned} \tag{4.8}$$

while for each integer $n_0(N)$ formulated in (2.24), thus by the inequalities (2.27) and (2.28), and since $M \geq n_0(N) \geq N \geq K+1$ so by (4.8) then it implies

$$\begin{aligned} & |f(s) - f_M(s)| \\ = & \ |\sum_{n=M+1}^{\infty}\frac{1}{2^{n+1}}\Delta_n(w_\alpha)| \\ < & \ \frac{1}{2^{\frac{M}{2}+2}} + \sum_{n=M+1}^{\infty}\frac{n^{\varepsilon}}{2^{n+1}}\{n_n(p-\alpha)^{n_n}(\log(n_n+1))^{n_n} + n + n^{-\varepsilon}|\Delta_n(w_p)|\} \\ < & \ \frac{1}{2^{\frac{N}{2}+1}} + \frac{1/2}{3^{(\log(N+1))^{\lambda_0}}} \ . \end{aligned} \tag{4.9}$$

Therefore by (4.5) with $f(s_1) = f(s)$ and by (4.9), then it implies

$$|f(s_1) - f_M(s)| = |f(s) - f_M(s)| < \frac{1}{2^{\frac{N}{2}+1}} + \frac{1/2}{3^{(\log(N+1))^{\lambda_0}}} \ . \tag{4.10}$$

Therefore by putting (4.7) and (4.10) with $M \geq N$ together, and by applying the triangular inequality then it implies

$$|f(s_1) - \tilde{f}_M(\tilde{s_1})| < \frac{1}{2^{N/2}} + \frac{1/2}{3^{(\log(N+1))^{\lambda_0}}} \ , \tag{4.11}$$

where $\tilde{s_1} \in K_s$ is the given compact subset, for which then by (4.6) it is the inverse image of the given compact sucbset K_c contained in $\tilde{\mathbf{C}}$. Therefore the statement (1) is proved. Meanwhile by (4.5) then it implies $f(s) = f(s_1)$, so by (4.11) it implies the statement (2). □

Similarly to the uniform approximations of Lemma 4.3, we will have the similar conclusion achieved for the difference polynomials $f_N(s_{\alpha,\beta})$ which is defined by

Definition 4.4. On the Riemann sub-surface $\tilde{\mathbf{S}}$, and for the points s_α and s_β defined in (4.4) which are contained in the compact subset K_s defined in (4.13), then we define the difference polynomials

$$f_M(s_{\alpha,\beta}) := \tilde{f}_M(s_\alpha) - \tilde{f}_M(s_\beta) \ , \quad s_\sigma = \exp(\frac{-(\sigma+it)}{Q_{M_0}}) \ , \tag{4.12}$$

where for each polynomial $\tilde{f}_M(s_\sigma)$ it is formulated in (4.1) by (4.4).

Lemma 4.5. *On any compact subset K_s of the Riemann surface $\tilde{\mathbf{S}}$ defined in* (4.3) *with its $-Q_{N_0}\log{(.)}$ mapping image K_c falls into the region $\tilde{\mathbf{C}}$. With assuming the same conditions of Lemma 4.3 and all the integers $M \geq \max\{n_0(N), N\}$. Then For the difference polynomials $f_M(s_{\alpha,\beta})$ defined in* (4.12) *which also enjoy the uniform approximating as those of the series $f_M(\alpha+it_0) - f_M(\beta+it_0)$ have, and approximate the series $f(\alpha+it_0) - f(\beta+it_0)$ defined in* (4.5) *within a perturbation of at most $2\vartheta(N)$ by the pull-back mapping $-Q_{M_0}\log(.)$ on the Riemann sub-surface $\tilde{\mathbf{S}}$ where the numbers $\vartheta(N)$ are defined in (4.6).*

Proof. By (4.12), then the difference $\{f_M(\alpha+it_0) - f_M(\beta+it_0)\} - f_M(s_{\alpha,\beta})$ equals $\{f_M(\alpha+it_0) - \tilde{f}_M(s_\alpha)\} - \{f_M(\beta+it_0) - \tilde{f}_M(s_\beta)\}$, and then we estimate twice all the estimations performed in proving Lemma 4.3. Therefore we have the similar conclusion as that of Lemma 4.3. □

Together with respect to Lemma 4.3 and Lemma 4.5, we specify their associated given compact subsets K_s and K_c by the following relation

$$-Q_{N_0}\log(.) : K_s \longrightarrow K_c = \{s \in \tilde{\mathbf{C}} \mid \alpha \leq Re(s) \leq \beta, \ Im(s) = t_0\} \ , \tag{4.13}$$

with the condition $0 < \alpha \leq \beta < 1$, and for each given integer $n_0(N)$ defined in the estimations formulated in (2.24), then we have the following locally uniform approximating theorem

Theorem 4.6. *On the given compact subset K_s with assuming the same conditions of Lemma 4.3, and for each Dirichlet η-function $f(s)$ which is defined in (3.1), and for all sufficiently large integers M with $M \geq \max\{n_0(N), N\}$, then for each M and for each number $\vartheta(N)$ which is defined in (4.6)*

(1) *The polynomial $\tilde{f}_M(\tilde{s_1})$ approximates the series $f(\sigma+it_0)$ within a perturbation of at most $\vartheta(N)$, and we denote it by*

$$\tilde{f}_M(\tilde{s_1}) \asymp f(\sigma+it_0) \ , \quad \tilde{s_1} = s_\sigma = \exp(\frac{-(\sigma+it)}{Q_{M_0}}) \quad \textit{within} \ \ \vartheta(N). \tag{4.14}$$

(2) *For the difference polynomial $f_M(s_{\alpha,\beta})$ formulated in (4.12), where $s_\sigma = \exp(\frac{-(\sigma+it)}{Q_{M_0}})$ is defined on K_s. Then it approximates the series $f(\alpha+it_0) - f(\beta+it_0)$ within a perturbation of at most $2\vartheta(N)$, and we denote it by*

$$f_M(s_{\alpha,\beta}) \asymp f(\alpha+it_0) - f(\beta+it_0) \quad \textit{within} \ \ 2\vartheta(N). \tag{4.15}$$

Now we consider the following formulations, which range from (4.16) to (4.20), as an alternative progression to succeed the formulation (3.6) of Section 3. For each given positive integer r

and for each Dirichlet η-function $f(s)$ defined in (3.1) then for each r-th derivative $f^{(r)}(s)$, we claim that we have the following locally uniform convergence

$$f^{(r)}(s) = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \sum_{k=0}^{n} \binom{n}{k} (-1)^k b_{k+1} (-1)^r (\log(k+1))^r (k+1)^{-s} \tag{4.16}$$
$$\text{on the given compact subset } K_c \text{ defined in (4.6).}$$

The reason for (4.16) is that, by the root test applying to the sequence of power series $\{\tilde{f}_M(\tilde{s_1}) \mid M \geq \max\{n_0(N), N\}\}$ defined in (4.1), then this sequence is locally uniformly convergent on the Riemann sub-surface $\{s_1 \in \mathbf{S} \mid 0 < |s_1| < 1\}$. Namely, with Lemma 4.3 then we have the following locally uniform convergence

$$f(s) = \lim_{\substack{|\log(k+1) - \frac{P_k}{Q_{N_0}}| < \frac{1}{N^2 2^{2N+2}} \\ 0 \leq k \leq N \\ N \to \infty}} \tilde{f}_N(\tilde{s_1}) \ , \quad -Q_{N_0} \log(\tilde{s_1}) = s \tag{4.17}$$
$$\text{on the given compact subset } K_c \text{ defined in (4.6),}$$

and by defining the following notation

$$\lim_{(N)} := \lim_{\substack{|\log(k+1) - \frac{P_k}{Q_{N_0}}| < \frac{1}{N^2 2^{2N+2}} \\ 0 \leq k \leq N \\ N \to \infty}} . \tag{4.18}$$

Then for each r-th derivative $f^{(r)}(s)$, we have the following locally uniform convergence

$$\begin{aligned} f^{(r)}(s) &= \lim_{(N)} \frac{d^r}{ds^r} (\tilde{f}_N(\tilde{s_1})) \ , \quad \tilde{s_1} = \exp(\frac{-s}{Q_{N_0}}) \\ &= \lim_{(N)} \sum_{k=0}^{N} (-1)^k b_{k+1} \sum_{l=k}^{N} \binom{l}{k} \frac{1}{2^{l+1}} \frac{d^r}{ds^r} \tilde{s_1}^{P_k} \\ &\quad \text{on the given compact subset } K_c \text{ defined in (4.6)} \\ &= \lim_{(N)} \tilde{s_1}^{-r} \sum_{k=0}^{N} (-1)^k b_{k+1} \sum_{l=k}^{N} \binom{l}{k} \frac{1}{2^{l+1}} (-1)^r (\frac{P_k}{Q_{N_0}})^r \tilde{s_1}^{P_k} \ , \end{aligned} \tag{4.19}$$

since the derivative

$$\frac{d^r}{ds^r} \tilde{s_1}^{P_k} = \frac{(-1)^r P_k(P_k - 1) \dots (P_k - r + 1)}{Q_{N_0}^r} \tilde{s_1}^{P_k - r} \ ,$$

and so long as all the rational numbers r_k satisfy $r_k = \frac{P_k}{Q_{N_0}} \to \log(k+1)$ simultaneously, then we have

$$\begin{aligned} (\frac{1}{Q_{N0}})^r \binom{P_k}{r} &= \frac{P_k(P_k - 1) \dots (P_k - r + 1)}{(Q_{N_0})^r r!} \\ &\longrightarrow \frac{(\log(k+1))^r}{r!} \ , \end{aligned} \tag{4.20}$$

which is a strictly increasing sequence of positive real numbers, and by the locally uniform limit $\tilde{s_1}^{-r} \to 1$ as $N \to \infty$ on the compact subsets. Therefore the formula (4.16) is proved.

Then similarly to the Dirichlet η-functions $f(s)$ defined in (3.1), thus for each function $f^{(r)}(s)$, first we have all of the similar definitions for which they are formulated from (1.5) to (1.16).

Therefore similarly to (2.1) we define

$$\Delta_n(\frac{b_\nu(-1)^r(\log(\nu))^r}{\nu^s}) := \sum_{k=0}^{n}\binom{n}{k}(-1)^k b_{k+1}(-1)^r(\log(k+1))^r(k+1)^{-s} \tag{4.21}$$
$$\text{where } b_{k+1}(\log(k+1))^r = o((k+1)^\varepsilon), \text{ for any } \varepsilon > 0 \ .$$

Thus for each finite difference (4.21) we also have the similar estimation as that of (2.18). Hence for each r-th derivative $f^{(r)}(s)$ we also have the similar Key lemmas as those of Section 2, and the similar locally uniform approximations as those previous results of this section.

Then for each function $f^{(r)}(s)$ interpreted in (4.19), it also satisfies the similar statement of Theorem 4.6. Namely by (4.16) and by the formulation similar to (4.1), then for the polynomials

$$\begin{aligned} \tilde{f_N^{(r)}}(\tilde{s_1}) :&= \sum_{n=0}^{N}\frac{1}{2^{n+1}}\sum_{k=0}^{N}\binom{n}{k}(-1)^k b_{k+1}(-1)^r(\log(k+1))^r\tilde{s_1}^{P_k} \\ &= \sum_{k=0}^{N}\sum_{l=k}^{N}\binom{l}{k}\frac{1}{2^{l+1}}(-1)^k b_{k+1}(-1)^r(\log(k+1))^r\tilde{s_1}^{P_k} \ , \\ f_N^{(r)}(s_{\alpha,\beta}) :&= \tilde{f_N^{(r)}}(s_\alpha) - \tilde{f_N^{(r)}}(s_\beta) \ , \quad \text{where } s_\sigma = \tilde{s_1} = \exp(\frac{-(\sigma+it)}{Q_{N_0}}) \ . \end{aligned} \tag{4.22}$$

And for the given integer $m_0 \geq 1$, then we have

Theorem 4.7. *On the given compact subset K_s defined in (4.6) and for all sufficiently large integers $M \geq \max\{M(K_s,m_0),N\}$, where the given and fixed integer $M(K_s,m_0)$ depends on K_s and on the functions $f^{(r)}(s)$ with all the integers r in $0 \leq r \leq m_0$, and where $f(s)$ is the Dirichlet η-function defined in (3.1) with each derivative $f^{(r)}(s)$, and with assuming the same conditions of Lemma 4.3, then for each M and for each number $\vartheta(N)$ which is defined in (4.6)*

1. *The polynomials $\tilde{f_M^{(r)}}(\tilde{s_1})$ approximate the series $f^{(r)}(\sigma+it_0)$ within a perturbation of at most $\vartheta(N)$, and we denote them by*

$$\tilde{f_M^{(r)}}(\tilde{s_1}) \asymp f^{(r)}(\sigma+it_0) \ , \ \tilde{s_1} = s_\sigma = \exp(\frac{-(\sigma+it)}{Q_{M_0}}) \quad \textit{within} \ \ \vartheta(N) \ . \tag{4.23}$$

2. *For the difference polynomials $f_M^{(r)}(s_{\alpha,\beta})$ formulated similarly as those of (4.12), where $s_\sigma = \exp(\frac{-(\sigma+it)}{Q_{M_0}})$ is defined on K_s. Then they approximate the series $f^{(r)}(\alpha+it_0) - f^{(r)}(\beta+it_0)$ within a perturbation of at most $2\vartheta(N)$, and we denote them by*

$$f_M^{(r)}(s_{\alpha,\beta}) \asymp f^{(r)}(\alpha+it_0) - f^{(r)}(\beta+it_0) \quad \textit{within} \ \ 2\vartheta(N) \ . \tag{4.24}$$

Proof. To choose all such sufficiently large integers M, first by all the arguments follow from the formulation of (4.21) there exists a sufficiently large integer M_r for each function $f^{(r)}(s)$ such that, for any given integer $M \geq M_r$ it satisfies the similar statements of Theorem 4.6 with $M_0 = n_0(N)$. Second we let the integer $M(K_s,m_0)$ be the maximum of the integers M_r, for all integers r in $0 \leq r \leq m_0$. So by all the arguments follow from the formulation of (4.21), then for all the functions $f^{(r)}(s)$ with $0 \leq r \leq m_0$ and for all the given integers $M \geq \max\{M(K_s,m_0),N\}$, altogether they satisfy the similar statements of Theorem 4.6. □

We note that: For all the integers $M \geq \max\{M(K_s,m_0),N\}$, then eventually $\max\{M(K_s,m_0),N\} = N$; because for each of their associated $M(K_s,m_0)$ it is a minimum integer which is designed for the asymptotics of Theorem 4.7 which is defined in the proof of Theorem 4.7; and because with regard to each given sufficiently large N, for which it can be traced back to its defined minimum $n_0(N)$ formulated in (2.24) namely, the choice of the minima $n_0(N)$ and

$M(K_s, m_0)$ are, indeed, independent of each given large integer N but depends on the given compact subset K_s; and for such a choice of the minimum $n_0(N)$ it satisfies the conditions of Lemma 4.3, hence of Lemma 2.2 where it is shown by two estimations explained in (2.24) which implies eventually $N \geq n_0(N)$, for which it is explained at the notice after Lemma 2.2; hence by definition of $M(K_s, m_0)$ which is also such a choice of the minimum defined in the proof of Theorem 4.7 and which extends the meaning of $n_0(N)$ for $f(s)$ to the same meaning of $M(K_s, m_0)$ for all $f^{(r)}(s)$ with $0 \leq r \leq m_0$ hence eventually $N \geq M(K_s, m_0)$. Hence for each integer N greater than its associated minima $M(K_s, m_0) \geq n_0(N)$ then we identify it with M.

5. Extended Riemann Hypothesis

On the given compact subset K_s defined in (4.6) which is contained in the Riemann subsurface $\tilde{\mathbf{S}}$, and for any two given points s_α and s_β contained in K_s by which, then they are defined in the following relations

$$-Q_{N_0}\log(s_\alpha) = \alpha + it_0 \quad \text{and} \quad -Q_{N_0}\log(s_\beta) = \beta + it_0 \ , \tag{5.1}$$

where the condition $0 < \alpha \leq \beta < 1$ is given. Furthermore, for any given nonnegative integer P_k, then we have the following relation

$$s_\alpha^{P_k} = \sum_{j=1}^{P_k} \binom{P_k}{j} (s_\alpha s_\beta^{-1} - 1)^j s_\beta^{P_k} + s_\beta^{P_k} \ . \tag{5.2}$$

Therefore for each difference polynomial $f_N(s_{\alpha,\beta}) = \tilde{f}_N(s_\alpha) - \tilde{f}_N(s_\beta)$ defined in (4.12), then by (5.2) it can be expressed as the following

$$\begin{aligned} &\sum_{n=1}^{P_N} (\beta-\alpha)^n \exp(\frac{-n\sigma_{\alpha,\beta}}{Q_{N_0}}) \\ &\qquad \sum_{k=0}^{N} \frac{1}{(Q_{N_0})^n} \binom{P_k}{n} (-1)^k b_{k+1} \exp(\frac{-(\beta+it_0)P_k}{Q_{N_0}}) \\ &\qquad\qquad \sum_{l=k}^{N} \binom{l}{k} \frac{1}{2^{l+1}} \ , \\ &\qquad \binom{P_k}{n} := 0 \quad \text{if } n > P_k \ , \end{aligned} \tag{5.3}$$

where the positive real numbers $\sigma_{\alpha,\beta}$ with $\alpha < \sigma_{\alpha,\beta} < \beta$, is derived from the Mean Valued Theorem applying to

$$\begin{aligned} \exp(\frac{-\alpha}{Q_{N_0}}) - \exp(\frac{-\beta}{Q_{N_0}}) &= (\exp(\frac{(\beta-\alpha)}{Q_{N_0}}) - 1)\exp(\frac{-\beta}{Q_{N_0}}) \\ &= (\frac{\beta-\alpha}{Q_{N_0}})\exp(\frac{-\sigma_{\alpha,\beta}}{Q_{N0}}) \ . \end{aligned} \tag{5.4}$$

Now for each pair of real numbers x_0 and y_0 with the condition $\alpha \leq x_0 \leq y_0 \leq \beta$, we first define the notation

$$w_\sigma := \sigma + it_0 \ ,$$

then for the given sufficiently large integer $M \geq \max\{n_0(N), N\}$ which depends on the compact subset K_c given in Theorem 4.6, we have the following asymptotic

$$\begin{aligned} &f(w_{x_0}) \asymp f_M(s_{x_0,y_0}) + f(w_{y_0}) \quad \text{within} \ \ 2\vartheta(N) \ , \\ &\vartheta(N) := \frac{1}{2^{N/2}} + \frac{1/2}{3^{(\log(N+1))^{\lambda_0}}} \ , \end{aligned} \tag{5.5}$$

where the function $f_M(s_{x_0,y_0})$ has the similar expression as that of $f_N(s_{\alpha,\beta})$ interpreted in (5.3).

Therefore by (5.3) and (5.5), then by Theorem 4.7 we have the following estimation

$$\begin{aligned} |\{f(w_{x_0}) - f(w_{y_0})\} - &\sum_{n=1}^{P_M}(y_0 - x_0)^n \exp(\frac{-n\sigma_{x_0,y_0}}{Q_{M_0}}) \\ &\sum_{k=0}^{M}\binom{P_k}{n}(-1)^k b_{k+1}\exp(\frac{-(y_0+it_0)P_k}{Q_{M_0}})\sum_{l=k}^{M}\binom{l}{k}\frac{1}{2^{l+1}}| \\ &< 2\vartheta(N) \ . \end{aligned} \tag{5.6}$$

And by (4.23) of Theorem 4.7 with $M \geq \max\{M(K_s, m_0), N\}$, and for the given integer $m_0 \geq 1$ then we have the locally uniform estimations

Lemma 5.1. *On the given compact subset K_s defined in (4.6) and for all sufficiently large integers $M \geq \max\{M(K_s, m_0), N\}$, where the integers $M(K_s, m_0)$ are formulated in Theorem 4.7 and with all the integers r in $0 \leq r \leq m_0$, where $f(s)$ is the Dirichlet η-function defined in (3.1) with each derivative $f^{(r)}(s)$. Then for the polynomials $\widetilde{f_M^{(r)}}(\tilde{s_1})$ defined in (4.22) with $\tilde{s_1} = s_{y_0}$, they together approximate the series $f^{(r)}(y_0 + it_0)$ within a perturbation of at most $\vartheta(N)$, and we denote each one of them by*

$$|\widetilde{f_M^{(r)}}(\tilde{s_1}) - f^{(r)}(y_0 + it_0)| < \vartheta(N) \ , \quad \tilde{s_1} = s_{y_0} = \exp(\frac{-(y_0 + it_0)}{Q_{M_0}}) \ . \tag{5.7}$$

Proof. Since (4.14) namely, for each given y_0 in $\alpha \leq y_0 \leq \beta$ and for each given M with $M \geq \max\{M(K_s, m_0), N\}$ where the integer $M(K_s, m_0)$ is formulated in Theorem 4.7, then

$$\widetilde{f_M^{(r)}}(\tilde{s_1}) \asymp f^{(r)}(y_0 + it_0) \ , \quad \tilde{s_1} = s_{y_0} \quad \text{within the perturbation } \vartheta(N) \ , \tag{5.8}$$

which implies (5.7). □

We observe that similarly to the argument of (2.12) and (2.13), we have

Lemma 5.2. *For the difference polynomial $f_N(s_{\alpha,\beta})$ defined in (5.3), in $\alpha = x_0$ and $\beta = y_0$, and with conditions of Lemma 5.1 then we have the following asymptotic formulas*

(1) *By (5.3) for each integer r in $1 \leq r \leq P_M$ and for each $M \geq N$*

$$\exp(\frac{-r\sigma_{x_0,y_0}}{Q_{M_0}})\frac{1}{(Q_{M_0})^r}\binom{P_k}{r} \asymp \frac{1}{r!}(\log(k+1))^r \ . \tag{5.9}$$

(2) *By substituting n for r in (5.9) then for each $M \geq N$*

$$\begin{aligned} f_M(s_{x_0,y_0}) \asymp \sum_{n=1}^{\infty}(y_0 - x_0)^n &\sum_{k=0}^{M}\frac{(\log(k+1))^n}{n!}(-1)^k b_{k+1} \\ &\exp(-(y_0+it_0)\log(k+1))\sum_{l=k}^{N}\binom{l}{k}\frac{1}{2^{l+1}} \ . \end{aligned} \tag{5.10}$$

We note that: By the argument of (2.13), the right hand side polynomial of (5.10) is a dominant asymptotic w.r.t. the left hand side difference polynomial $f_N(s_{x_0,y_0})$. Namely the domination is decided by comparing all the absolute values of each pair of the corresponding coefficients interpreted in (5.10) w.r.t the same monomial term $(y_0 - x_0)^n$.

Lemma 5.3. *With assuming the conditions of Lemma 5.1, and in addition to the uniform approximations of the numbers* $\log(k+1)$ *with the integers* k *in* $0 \le k \le M$ *which are formulated in (1.5) and (1.6), we add the following condition to the integers* Q_{M_0} *such that on* K_s

$$|\exp(\frac{-\sigma_{x_0,y_0}}{Q_{M_0}}) - 1| < \frac{1}{(m_0+1)M^2 2^{2M+2}} . \tag{5.11}$$

Then for each integer r *in* $1 \le r \le m_0$ *and by the formula (5.9), it implies the following estimation*

$$\begin{aligned} &|\exp(\frac{-r\sigma_{x_0,y_0}}{Q_{M_0}})\frac{P_k(P_k-1)\dots(P_k-r+1)}{Q_{M_0}^r} - (\log(k+1))^r| \\ &\qquad < \frac{(r+1)(\log(M))^r}{M^2 2^{2M+2}} ; \end{aligned} \tag{5.12}$$

for $\tilde{s_1} := s_{y_0} = \exp(-(y_0+it_0)/Q_{M_0})$*, by (5.10) it implies the following uniform estimation*

$$\begin{aligned} &|\exp(\frac{-r\sigma_{x_0,y_0}}{Q_{M_0}})\sum_{k=0}^{M}\frac{1}{(Q_{M_0})^r}\binom{P_k}{r}(-1)^k b_{k+1}\exp(\frac{-(y_0+it_0)P_k}{Q_{M_0}}) \\ &\qquad \sum_{l=k}^{M}\binom{l}{k}\frac{1}{2^{l+1}} - \frac{(-1)^r}{r!}f_M^{(r)}(\tilde{s_1})| \\ &\qquad < \frac{1}{r!}\frac{(r+1)(\log(M))^r}{2^{2M+2}} . \end{aligned} \tag{5.13}$$

Proof. We let

$$\begin{aligned} a_0 &:= \exp(\frac{-r\sigma_{x_0,y_0}}{Q_{M_0}}) , \\ a_j &:= (\log(k+1))^{j-1}\frac{\prod_{i=j-1}^{r-1}(P_k+i-j+1-r+1)}{Q_{M_0}^{r-(j-1)}} , \quad 1 \le j \le r , \end{aligned} \tag{5.14}$$

and apply the triangular inequality to estimate the following equality

$$\begin{aligned} &|a_0a_1 - (\log(k+1))^r| \\ &\qquad = |a_0a_1 - a_1 + \sum_{j=1}^{r-1}(a_j - a_{j+1}) + a_r - (\log(k+1))^r| , \end{aligned} \tag{5.15}$$

for which, before estimating (5.12), it requires the following r-many estimations

$$\begin{aligned} &|\frac{P_k-\ell}{Q_{M_0}} - \log(k+1)| , \quad 0 \le \ell \le r-1 \\ &\qquad < |\frac{(P_k-\ell-P_k)}{Q_{M_0}}| + |\frac{P_k}{Q_{M_0}} - \log(k+1)| \\ &\qquad < \frac{2r}{(m_0+1)M^2 2^{2M+2}} + \frac{1}{M^2 2^{2M+2}} , \end{aligned}$$

which is by (5.11) with applying the Mean Valued Theorem to the exponential function exp(.), with real numbers x_0, y_0 restricted to $0 < \alpha \leq x_0 \leq y_0 \leq \beta < 1$, hence we may assume

$$|\frac{1}{Q_{M_0}}| < \frac{2}{(m_0+1)M^2 2^{2M+2}} \ , \tag{5.16}$$

which can be chosen, since whenever in the numeric approximation (1.6) we can always choose a sufficiently large integer Q_{M_0} to meet both (1.6) and the requirement (5.16). Hence initiated firstly by (5.9), then by (5.14) and (5.15), and by (5.16) hence we have the estimation (5.12).

For the estimation (5.13) which is suggested by (5.10), it follows from (4.22) that as for the function $\tilde{f_M^{(r)}}(\tilde{s_1}) = \tilde{f_M^{(r)}}(s_{y_0})$ where $\tilde{s_1} := s_{y_0} = \exp(-(y_0+it_0)/Q_{M_0})$. So it exactly means that the following functions

$$\begin{aligned}\exp(\frac{-r\sigma_{x_0,y_0}}{Q_{M_0}})\sum_{k=0}^{M}\frac{1}{(Q_{M_0})^r}\binom{P_k}{r}(-1)^k b_{k+1}\exp(\frac{-(y_0+it_0)P_k}{Q_{M_0}})& \\ \sum_{l=k}^{M}\binom{l}{k}\frac{1}{2^{l+1}}& \ ,\end{aligned} \tag{5.17}$$

approximate the function $\frac{(-1)^r}{r!}\tilde{f_M^{(r)}}(s_{y_0})$ in sense of the approximations (1.5) and (1.6), for which it is now placed in estimating the left hand side of the inequality (5.13). So by applying the triangular inequality with estimation (5.12) and by estimation (4.2) of Lemma 4.1, then we have the locally uniform estimation (5.13) defined on the compact subset K_s. □

Further both (5.7) of Lemma 5.1 and (5.13) of Lemma 5.3 motivate us to define the following functions in the variables x_0 and y_0, which is purposedly suggested by the decomposition (5.20) for the difference $f_M(s_{x_0,y_0})$:

Definition 5.4. For each integer r in $1 \leq r \leq m_0$ we define the function

$$\begin{aligned}\Gamma(x_0,y_0,r) := \exp(\frac{-r\sigma_{x_0,y_0}}{Q_{M_0}})\sum_{k=0}^{M}\frac{1}{Q_{M_0}^r}\binom{P_k}{r}(-1)^k b_{k+1}& \\ \exp(\frac{-(y_0+it_0)P_k}{Q_{M_0}})\sum_{l=k}^{M}\binom{l}{k}\frac{1}{2^{l+1}}& \ ,\end{aligned} \tag{5.18}$$

such that for the polynomial $f_M(s_{x_0,y_0})$, its coefficient of the degree r-th term associated with the monomial $(y_0-x_0)^r$ is $\Gamma(x_0,y_0,r)$.

$$\begin{aligned}\Omega(x_0,y_0,m_0) := \sum_{n=m_0+1}^{P_M}(y_0-x_0)^n\exp(\frac{-n\sigma_{x_0,y_0}}{Q_{M_0}})\sum_{k=0}^{M}\frac{1}{Q_{M_0}^n}\binom{P_k}{n}(-1)^k b_{k+1}& \\ \exp(\frac{-(y_0+it_0)P_k}{Q_{M_0}})\sum_{l=k}^{M}\binom{l}{k}\frac{1}{2^{l+1}}& \ ,\end{aligned} \tag{5.19}$$

hence we have the decomposition

$$f_M(s_{x_0,y_0}) = \sum_{r=1}^{m_0-1}(y_0-x_0)^r\Gamma(x_0,y_0,r) + \Omega(x_0,y_0,m_0) \ . \tag{5.20}$$

Theorem 5.5. *With assuming the condition of Lemma 5.3, and for each integer M with $M \geq \max\{M(K_s,m_0),N\}$ which is formulated in Lemma 5.1. Then for each integer r in $1 \leq r \leq m_0$*

and for each real number δ *in* $0 < \alpha \leq x_0 \leq \delta \leq y_0 \leq \beta < 1$ *and for each number*

$$\vartheta(N) := \frac{1}{2^{N/2}} + \frac{1/2}{3^{(\log(N+1))^{\lambda_0}}},$$

we have the locally uniform inequalities

(1) *For the r-th derivative* $f^{(r)}(s)$ *of the Dirichlet* η*-function* $f(s)$ *we have the locally uniform estimation i.e., for* $\tilde{s_1} := s_\delta = \exp(-(\delta + it_0)/Q_{M_0})$ *on* K_s*, it implies*

$$\begin{aligned} &|\exp(\frac{-r\sigma_{x_0,\delta}}{Q_{M_0}})\sum_{k=0}^{M}\frac{1}{(Q_{M_0})^r}\binom{P_k}{r}(-1)^k b_{k+1}\exp(\frac{-(\delta+it_0)P_k}{Q_{M_0}}) \\ &\qquad\qquad \sum_{l=k}^{M}\binom{l}{k}\frac{1}{2^{l+1}} \;-\; \frac{(-1)^r}{r!}f^{(r)}(\delta+it_0)\;| \\ &< \;\frac{1}{r!}\frac{(r+1)(\log(M))^r}{2^{2M+2}} + \frac{1}{r!}\vartheta(N)\;, \end{aligned} \tag{5.21}$$

and in decomposition (5.20) for $f^{(r)}(s)$ *we substitute* y_0 *for* x_0*, and* δ *for* y_0

$$\begin{aligned} &|\exp(\frac{-r\sigma_{y_0,\delta}}{Q_{M_0}})\sum_{k=0}^{M}\frac{(-1)^r}{(Q_{M_0})^r}\binom{P_k}{r}(-1)^k b_{k+1}\exp(\frac{-(\delta+it_0)P_k}{Q_{M_0}}) \\ &\qquad\qquad \sum_{l=k}^{M}\binom{l}{k}\frac{1}{2^{l+1}} \;-\; \frac{1}{r!}f^{(r)}(\delta+it_0)\;| \\ &< \;\frac{1}{r!}\frac{(r+1)(\log(M))^r}{2^{2M+2}} + \frac{1}{r!}\vartheta(N)\;. \end{aligned} \tag{5.22}$$

(2) *By the decomposition (5.20) of* $f_M^{(r)}(s_{x_0,y_0})$ *with substituting* δ *for* y_0*, and with Theorem 4.7 applying on* $f^{(r)}(s)$ *then it implies*

$$\begin{aligned} &|f^{(r)}(x_0+it_0) - f^{(r)}(\delta+it_0) - \sum_{n=1}^{m_0-r}(\delta-x_0)^n\frac{(-1)^n}{n!}f^{(n+r)}(\delta+it_0)| \\ &< 2\vartheta(N) + \sum_{n=1}^{m_0-r}(\delta-x_0)^n\{\frac{1}{n!}\frac{(n+r+1)(\log(M))^{n+r}}{2^{2M+2}} + \vartheta(N)\} + |\Omega(x_0,\delta,m_0)|, \end{aligned} \tag{5.23}$$

similarly in decomposition (5.20) for $f^{(r)}(s)$ *we substitute* y_0 *for* x_0*, and* δ *for* y_0

$$\begin{aligned} &|f^{(r)}(y_0+it_0) - f^{(r)}(\delta+it_0) - \sum_{n=1}^{m_0-r}(y_0-\delta)^n\frac{1}{n!}f^{(n+r)}(\delta+it_0)| \\ &< 2\vartheta(N) + \sum_{n=1}^{m_0-r}(y_0-\delta)^n\{\frac{1}{n!}\frac{(n+r+1)(\log(M))^{n+r}}{2^{2M+2}} + \vartheta(N)\} + |\Omega(y_0,\delta,m_0)|. \end{aligned} \tag{5.24}$$

Proof. We combine both (5.7) of Lemma 5.1 and (5.13) of Lemma 5.3 with applying triangular inequality, then by decomposition (5.20) we have proved (5.21) and (5.22). For the uniform estimation (5.23), first by decomposition (5.20) in $f_M^{(r)}(s_{x_0,\delta})$ for $f^{(r)}(s)$ and by Theorem 4.7

$$|f^{(r)}(x_0+it_0) - f^{(r)}(\delta+it_0) - f_M^{(r)}(s_{x_0,\delta})| < 2\vartheta(N)\;. \tag{5.25}$$

Second with (5.21) we apply the triangular inequality to the following inequality (5.26) which is a translation from (5.25)

$$\begin{aligned}
&|f^{(r)}(x_0+it_0) - f^{(r)}(\delta+it_0) - \\
&\sum_{n=1}^{P_M}(\delta - x_0)^n \exp(\frac{-n\sigma_{x_0,\delta}}{Q_{M_0}}) \sum_{k=0}^{M} \frac{1}{(Q_{M_0})^n}\binom{P_k}{n}(-1)^k b_{k+1}(-1)^r(\log(k+1))^r \\
&\qquad \exp(\frac{-(\delta+it_0)P_k}{Q_{M_0}})\sum_{l=k}^{M}\binom{l}{k}\frac{1}{2^{l+1}}| \ < \ 2\vartheta(N)\ ,
\end{aligned} \tag{5.26}$$

that is, via the similar argument of (5.14) and (5.15), and (5.16) in proving the estimation (5.12) of Lemma 5.3 hence we have proved the locally uniform estimation (5.23). By the similar argument in combining (5.22) with decomposition (5.20) for $f^{(r)}(s)$, for which then it is similar to the proof of (5.23), hence we prove (5.24). □

Now we recall the works of Riemann and Hecke in which they have shown the functional equations $f(s) = \gamma(s)f(1-s)$ of their cases in the Dirichlet series considered, hence by that then we conclude that for the $\gamma(s)$-factors of the Riemann zeta function $\zeta(s)$ and the Dedekind zeta functions $\zeta_K(s)$ which we include in Definition 3.1, they have neither zeros nor poles on the critical strip $0 < Re(s) < 1$. Thus for such Dirichlet series $D(s)$ with the associated Dirichlet η-functions $f(s)$ defined in Definition 3.1, we naturally conclude that: For any given pair of the complex numbers $\alpha + it_0$ and $1 - \alpha + it_0$ with $0 < \alpha < 1$, we have either $f(\alpha + it_0) = f(1-\alpha+it_0) = 0$ or $f(\alpha+it_0)f(1-\alpha+it_0) \neq 0$.

Therefore we come to assume that for any such complex point $\alpha+it_0$ which is one of the zeros of the given Dirichlet η-function $f(s)$ and we formulate it as:

$$f(\alpha+it_0) = f(1-\alpha+it_0) = 0\ ,\quad \text{with } 0<\alpha\le 1/2\ . \tag{5.27}$$

Then we apply the Mean Valued Theorem and the Intermediate Valued Theorem to the real-valued functions $Re(f(s))$ and $Im(f(s))$, so that we are able to locate the positions of the two positive real numbers x_0 and y_0 such that, say for $Re(f(s))$

$$Re(f(x_0+it_0)) = Re(f(y_0+it_0))\ ,\quad \text{with } 0<\alpha\le x_0\le y_0\le 1-\alpha<1\ , \tag{5.28}$$

with the real value $(y_0 - x_0) > 0$ as small as we will.

We let δ_0+it_0 be the unique local extremum, between the chosen x_0+it_0 and y_0+it_0 which are defined in (5.28) with the real value $(y_0-x_0) > 0$ as small as we will, for which it is set for either the real-valued function $Re(f(s))$ or the real-valued function $Im(f(s))$ on the line segment $L := \{s = \sigma + it_0 \in \mathbb{C} \mid \alpha \le \sigma \le 1-\alpha\}$, where $f(s) = Re(f(s)) + iIm(f(s))$. With the assumption (5.27) then with $Re(f(s))$ we define

Definition 5.6. For the given positive integer m_0, we define it to be the index of the first nonvanishing coefficient $\frac{1}{m_0!}Re(f^{(m_0)}(\delta_0+it_0))$ of the Talyor series expansion of $Re(f(s))$ in $s \in L$ with respect to the point $\delta_0 + it_0$. Therefore for those real numbers y_0 which are defined in (5.28), the first nonvanishing coefficients of the Taylor series $Re(f(y_0+it_0)) - Re(f(\delta_0+it_0))$ is the real number $\frac{1}{m_0!}Re(f^{(m_0)}(\delta_0+it_0))$ and m_0 is an even positive integer, since δ_0+it_0 is the chosen local extremum.

Moreover, for the number $n_{\alpha,\beta} := M^2(\log M)^{n_0}$, and the integer $n_0 := [(\log M)^2] + 1$ which are defined in Lemma 5.10 at the below, we restrict the chosen real numbers x_0 and y_0, defined in (5.28), to satisfy

$$\frac{1}{2}(\frac{1}{n_{\alpha,\beta}})^2 < (y_0-x_0) < (\frac{1}{n_{\alpha,\beta}})^2\ ,\quad x_0<\delta_0<y_0\ . \tag{5.29}$$

Definition 5.7. For each integer q in $1 \leq q \leq m_0 - r$, r in $1 \leq r \leq m_0 - 1$ we define the function

$$\begin{aligned}\Gamma'(x_0, y_0, q) := \exp(\frac{-q\sigma_{x_0,y_0}}{Q_{M_0}}) \sum_{k=0}^{M} \frac{1}{Q_{M_0}^q} \binom{P_k}{q} (-1)^k b_{k+1} (-1)^r (\log(k+1))^r \\ \exp(\frac{-(y_0 + it_0)P_k}{Q_{M_0}}) \sum_{l=k}^{M} \binom{l}{k} \frac{1}{2^{l+1}} ,\end{aligned} \tag{5.30}$$

such that for the polynomial $f_M^{(r)}(s_{x_0,y_0})$, its coefficient of the degree q-th term associated with the monomial $(y_0 - x_0)^q$ is $\Gamma'(x_0, y_0, q)$.

$$\begin{aligned}\Omega'(x_0, y_0, m_0 - r) := \sum_{n=m_0-r+1}^{P_M} (y_0 - x_0)^n \exp(\frac{-n\sigma_{x_0,y_0}}{Q_{M_0}}) \sum_{k=0}^{M} \frac{1}{Q_{M_0}^n} \binom{P_k}{n} (-1)^k b_{k+1} \\ (-1)^r (\log(k+1))^r \exp(\frac{-(y_0 + it_0)P_k}{Q_{M_0}}) \sum_{l=k}^{M} \binom{l}{k} \frac{1}{2^{l+1}} ,\end{aligned} \tag{5.31}$$

hence we have the decomposition

$$f_M^{(r)}(s_{x_0,y_0}) = \sum_{q=1}^{m_0-r} (y_0 - x_0)^q \Gamma'(x_0, y_0, q) + \Omega'(x_0, y_0, m_0 - r) . \tag{5.32}$$

Theorem 5.8. *For the function $\Gamma(x_0, y_0, m_0)$ which is defined in Definition 5.4 with each integer $M \geq \max\{M(K_s, m_0), N\}$ for which we assume the conditions of Lemma 5.1 and Lemma 5.3, and by Definition 5.6 and (5.29). Then for each number*

$$\vartheta(N) := \frac{1}{2^{N/2}} + \frac{1/2}{3^{(\log(N+1))^{\lambda_0}}},$$

we have the locally uniform estimations

(1) *The locally uniform convergence limit on K_s with the perturbation $n_{\alpha,\beta}^{-2}|f^{(m_0+1)}(\delta_0 + it_0)| + 2\vartheta(N)$*

$$\lim_{\substack{|\log(k+1) - \frac{P_k}{Q_{M_0}}| < \frac{1}{M^2 2^{2M+2}} \\ 0 \leq k \leq M \\ N \to \infty,\ M \to \infty}} Re(\Gamma(x_0, y_0, m_0)) = \frac{(-1)^{m_0}}{m_0!} Re(f^{(m_0)}(\delta_0 + it_0)) \neq 0 . \tag{5.33}$$

(2) *The following locally uniform estimation for each function $\Omega(x_0, y_0, u)$ with integer u in $1 \leq u \leq m_0$, which is defined in Definition 5.4*

$$| \Omega(x_0, y_0, u) | < \frac{(y_0 - x_0)^{m_0}}{n_{\alpha,\beta}^2} , \tag{5.34}$$

where for the number $n_{\alpha,\beta} := M^2(\log M)^{n_0}$, and the integer $n_0 := [(\log M)^2] + 1$ which are defined in Lemma 5.10 at the below.

(3) *For each integer r in $0 \leq r \leq m_0 - 1$, exactly the same locally uniform estimation inequalities of (5.34) which also work with $|\Omega'(x_0, y_0, m_0 - r)|$, where $\Omega'(x_0, y_0, m_0 - r)$ and $\Omega(x_0, y_0, m_0 - r)$ are defined in Definition 5.7 and Definition 5.4 respectively.*

(4) *By (5.23) and (5.34) then for each integer r in $1 \leq r \leq m_0 - 1$ we have the following locally uniform estimation*

$$|Re(f^{(r)}(x_0 + it_0)) - (\delta_0 - x_0)^{m_0 - r} \frac{(-1)^{m_0 - r}}{(m_0 - r)!} Re(f^{(m_0)}(\delta_0 + it_0))| < 3\vartheta(N) + 2\frac{(y_0 - x_0)^{m_0}}{n_{\alpha,\beta}^2}. \tag{5.35}$$

Proof. Before go into proving, let us recall the formulation (4.16) for the function $f^{(r)}(s)$ and its successive explanations from (4.17) to (4.20). Then we will expect the result that the formulation in Definition 5.6 for each function $Re(f^{(r)}(s))$, in which the value $Re(f^{(m_0)}(\delta_0+it_0))$ will first be recovered in the limit presented at below by the functions $Re(\Gamma(x_0,\delta_0,m_0))$, for which it is defined in Definition 5.4 to be the coefficient of the degree m_0-th term for the polynomial $Re(f_M(s_{x_0,\delta_0}))$ which is expanded in a polynomial in the monomial $(\delta_0-x_0)^m$-term for $m=1,2,3,...,$ etc. by which we surely have the following limit

$$\lim_{\substack{|\log(k+1)-\frac{P_k}{Q_{M_0}}|<\frac{1}{M^2 2^{2M+2}} \\ 0\le k\le M \\ N\to\infty\ ,M\to\infty}} |Re(\Gamma(x_0,\delta_0,m_0))-\frac{(-1)^{m_0}}{m_0!}Re(f^{(m_0)}(\delta_0+it_0))|=0\ .$$

With this survey thus our proof of (5.33) is initially presented in (5.21) of Theorem 5.5 with $\delta=\delta_0$, thus by Definition 5.4 then in the following we will have the proof of (5.33).

So for each integer r in $1\le r\le m_0$ and each $\Gamma(x_0,y_0,r)$ which is defined in Definition 5.4 then by triangular inequality we claim

$$\begin{aligned}
&|\Gamma(x_0,y_0,r)-\frac{(-1)^r}{r!}f^{(r)}(\delta_0+it_0)| \\
&<|\Gamma(x_0,y_0,r)-\frac{(-1)^r}{r!}\tilde{f_M^{(r)}}(s_{x_0})|+|\frac{(-1)^r}{r!}\tilde{f_M^{(r)}}(s_{x_0})-\frac{(-1)^r}{r!}f^{(r)}(\delta_0+it_0)| \\
&<2\vartheta(N)\ +\ \frac{1}{r!}|f^{(r)}(x_0+it_0)-f^{(r)}(\delta_0+it_0)|\ ,
\end{aligned} \tag{5.36}$$

where $s_{x_0}=\exp(\frac{-(x_0+it_0)}{Q_{M_0}})$. The reason for (5.36) is that since for all integers r in $1\le r\le m_0$ and for each function $\Gamma(x_0,y_0,r)$ defined in Definition 5.4, then by (5.13) of Lemma 5.3 we have the locally uniform inequalies

$$|\Gamma(x_0,y_0,r)-\frac{(-1)^r}{r!}\tilde{f_M^{(r)}}(s_{x_0})|<\frac{1}{r!}\frac{(r+1)(\log(M))^r}{2^{2M+2}}<\frac{1}{2^{N/2}}\ .$$

While by (4.23) of Theorem 4.7 with whose locally uniform inequalities

$$|\tilde{f_M^{(r)}}(s_{x_0})-f^{(r)}(x_0+it_0)|<\vartheta(N)\ ,\ s_{x_0}=\exp(\frac{-(x_0+it_0)}{Q_{M_0}})\ ,$$

then by the triangular inequality it implies

$$\begin{aligned}
&\frac{1}{r!}\ |\tilde{f_M^{(r)}}(s_{x_0})-f^{(r)}(\delta_0+it_0)|\ ,\quad 0<\delta_0-x_0<n_{\alpha,\beta}^{-2}=(M^2(\log M)^{n_0})^{-2}\ , \\
&<\ \frac{1}{r!}|\tilde{f_M^{(r)}}(s_{x_0})-f^{(r)}(x_0+it_0)|\ +\ \frac{1}{r!}|f^{(r)}(x_0+it_0)-f^{(r)}(\delta_0+it_0)| \\
&<\ \frac{1}{r!}\vartheta(N)\ +\ \frac{1}{r!}|f^{(r)}(x_0+it_0)-f^{(r)}(\delta_0+it_0)|\ .
\end{aligned} \tag{5.37}$$

Then by (5.37) and the application of the triangular inequality it implies the inequality (5.36), in which for the case $r=m_0$ then $M\to\infty$, $x_0\to\delta_0$ and $y_0\to\delta_0$ hence both x_0+it_0 and y_0+it_0 tend to δ_0+it_0 the chosen local extremum of $Re(f(s))$. Thus we firstly prove the pointwise convergence limit (5.33). Secondly the limit (5.33) is a locally uniform convergence whose reason is due to the locally uniform estimations: First substituting integer m_0+1 for the integer m_0 and integer m_0 for the integer r in (5.23) of Theorem 5.5, then by applying inequality (5.36) with $r=m_0$.

For the proof of (5.34), firstly for each sufficiently large integer $M\ge\max\{M(K_s\ ,m_0),N\}$, where such integer $M(K_s,m_0)$ depends on K_s defined in (4.6) and for each function $f^{(r)}(s)$ with

r in $0 \leq r \leq m_0$, then we have the following decomposition: For the difference $f_M^{(r)}(s_{x_0,y_0})$ with r in $0 \leq r \leq m_0 - 1$

$$f_M^{(r)}(s_{x_0,y_0}) = \sum_{q=1}^{m_0-r} (y_0 - x_0)^q \Gamma'(x_0, y_0, q) + \Omega'(x_0, y_0, m_0 - r) \ ,$$

which is exactly the formula (5.32) of Definition 5.7 when r is in $1 \leq r \leq m_0 - 1$, while with $r = 0$, $\Omega'(x_0, y_0, m_0) = \Omega(x_0, y_0, m_0)$.

Secondly, since the Dirichlet η-function $f(s) := \sum_{n=1}^{\infty}(-1)^n b_n/n^s$, hence for each integer n in $m_0 < n \leq P_M$, then by Lemma 4.1 and Stirling's formula we have the following estimation

$$| \frac{(-1)^{k+r} b_{k+1} (\log(k+1))^{n+r}}{n!} \sum_{l=k}^{M} \binom{l}{k} \frac{1}{2^{l+1}} | < M^{1+\epsilon} \ , \tag{5.38}$$

for any given k in $0 \leq k \leq M$, and for any given $\epsilon > 0$.

Thirdly, so long as all the rational numbers r_k satisfy $r_k = \frac{P_k}{Q_{M_0}} \rightarrow \log(k+1)$ simultaneously for all k in $0 \leq k \leq M$, then it implies

$$\begin{aligned}(\frac{1}{Q_{M_0}})^n \binom{P_k}{n} &= \frac{P_k(P_k-1)\dots(P_k-n+1)}{(Q_{M_0})^n n!} \\ &\longrightarrow \frac{(\log(k+1))^n}{n!} \ ,\end{aligned} \tag{5.39}$$

for which, they are strictly increasing sequences of positive real numbers. Hence for each chosen large integer Q_{M_0} defined in (1.5) and (1.6), and by the limit defined in (5.39) then we have the following inequality

$$\frac{1}{(Q_{M_0})^n} \binom{P_k}{n} < \frac{(\log(k+1))^n}{n!} \ . \tag{5.40}$$

Now we employ the inequalities (5.38) and (5.40) to estimate the magnitude of the absolute values we claim in (5.34), for which it is first by the choice of the number $n_{\alpha,\beta} = M^2(\log M)^{n_0}$ which is defined in Lemma 5.10, such that for each integer n with $n \geq m_0 + 1 \geq 2$ then by (5.29) and Definition 5.7

$$\begin{aligned}&|(\frac{y_0 - x_0}{Q_{M_0}})^n \exp(\frac{-n\sigma_{x_0,y_0}}{Q_{M_0}}) \sum_{k=0}^{M} \binom{P_k}{n} (-1)^n b_n (-1)^r (\log(n+1))^r \\ &\qquad \exp(\frac{-(y_0 + it_0)P_k}{Q_{M_0}}) \sum_{l=k}^{M} \binom{l}{k} \frac{1}{2^{l+1}}| \\ &< \frac{(y_0 - x_0)^{m_0}}{n_{\alpha,\beta}} (\frac{1}{n_{\alpha,\beta}^2})^{n-1} \ .\end{aligned} \tag{5.41}$$

By (5.41) for all the n in $n \geq m_0 + 1 \geq 2$, then for each r in $0 \leq r \leq m_0 - 1$ we have the locally uniform estimation

$$| \Omega'(x_0, y_0, m_0) | < \frac{(y_0 - x_0)^{m_0}}{n_{\alpha,\beta}} \sum_{k=1}^{\infty} (\frac{1}{n_{\alpha,\beta}^2})^k < \frac{(y_0 - x_0)^{m_0}}{n_{\alpha,\beta}^2} \ , \tag{5.42}$$

while with $r = 0$, $\Omega'(x_0, y_0, m_0) = \Omega(x_0, y_0, m_0)$. Since this remark, hence by inequality (5.42) we prove firstly the estimation (5.34) when $r = 0$ we substitute u for m_0 and, secondly the statement (3) when we substitute $m_0 - r$ for m_0.

To prove (5.35), firstly we consider the finite difference $f_M^{(r)}(s_{x_0,\delta_0})$ in the locally uniform approximation of Theorem 4.7, then by (5,23) it implies

$$
\begin{aligned}
&|f^{(r)}(x_0+it_0) - f^{(r)}(\delta_0+it_0) - \sum_{n=1}^{m_0-r}(\delta_0-x_0)^n\frac{(-1)^n}{n!}f^{(n+r)}(\delta_0+it_0)| \\
&< 2\vartheta(N) + \sum_{n=1}^{m_0-r}(\delta_0-x_0)^n\{\frac{1}{n!}\frac{(n+r+1)(\log(M))^{n+r}}{2^{2M+2}} + \vartheta(N)\} + |\Omega'(x_0,\delta_0,m_0-r)|.
\end{aligned} \tag{5.43}
$$

And since $Re(f^{(r)}(\delta_0+it_0)) = 0$ for all r in $1 \le r \le m_0-1$, hence by (5.43) and Statement (3)

$$
\begin{aligned}
&|Re(f^{(r)}(x_0+it_0)) - (\delta_0-x_0)^{m_0-r}\frac{(-1)^{m_0-r}}{(m_0-r)!}Re(f^{(m_0)}(\delta_0+it_0))| \\
&\qquad < 3\vartheta(N) + 2|\Omega(x_0,\delta_0,m_0-r)| < 3\vartheta(N) + 2\frac{(y_0-x_0)^{m_0}}{n_{\alpha,\beta}^2}\ ,
\end{aligned} \tag{5.44}
$$

for any sufficiently large integers $M \ge \max\{M(K_s,m_0),N\}$. This proves (5.35). □

Theorem 5.9 (Extended Riemann Hypothesis)**.** *For all the Dirichlet η-functions $f(s)$ of the Riemann zeta function $\zeta(s)$ and the Dedekind zeta functions $\zeta_K(s)$ which are denoted as the Dirichlet series $D(s)$ defined in (3.1), while assuming the truth of Lemma 5.10. Then for $f(s)$ and $f^{(k)}(s)$ the k-th derivative of $f(s)$, and for $D(s)$ and $D^{(k)}(s)$ the k-th derivative of $D(s)$, all of their nontrivial zeros are contained in the vertical line $Re(s) = 1/2$.*

Proof. We assume the condition of Theorem 5.8. Firstly by (5.20) of Definition 5.4, then for the difference $Re(f_M(s_{y_0,x_0}))$ we have the decomposition

$$
Re(f_M(s_{y_0,x_0})) = \sum_{r=1}^{m_0}(x_0-y_0)^r Re(\Gamma(y_0,x_0,r)) + Re(\Omega(y_0,x_0,m_0))\ . \tag{5.45}
$$

So by the locally uniform estimation (5.6) and Definition 5.6 then we have

$$
|Re(f_M(s_{y_0,x_0}))| < 2\vartheta(N)\ ,\ \vartheta(N) := \frac{1}{2^{N/2}} + \frac{1/2}{3^{(\log(N+1))^{\lambda_0}}}\ . \tag{5.46}
$$

By applying the triangular inequality to estimate the inequality (5.46), then by (5.34) we have the locally uniform inequality

$$
\begin{aligned}
&-|\sum_{r=1}^{m_0-1}(x_0-y_0)^r Re(\Gamma(y_0,x_0,r))| \\
&\qquad + |(x_0-y_0)^{m_0}Re(\Gamma(y_0,x_0,m_0))| - \frac{|x_0-y_0|^{m_0}}{n_{\alpha,\beta}^2} < 2\vartheta(N),
\end{aligned} \tag{5.47}
$$

where the number $n_{\alpha,\beta} := M^2(\log M)^{n_0}$, and the integer $n_0 := [(\log M)^2]+1$ which are defined in Lemma 5.10 at the below.

Secondly by (5.18) and (5.21) with the condition (5.16), then (5.47) becomes

$$
\begin{aligned}
&-|\sum_{r=1}^{m_0-1}(x_0-y_0)^r\{\frac{(-1)^r}{r!}Re(f^{(r)}(x_0+it_0)) \pm \frac{1}{r!}\frac{(r+1)(\log(M))^r}{2^{2M+2}} \pm \frac{1}{r!}\vartheta(N)\}| \\
&\qquad + |(x_0-y_0)^{m_0}Re(\Gamma(y_0,x_0,m_0))| - \frac{|x_0-y_0|^{m_0}}{n_{\alpha,\beta}^2} < 2\vartheta(N)\ .
\end{aligned} \tag{5.48}
$$

By (5.35) and since $Re(f^{(r)}(\delta_0+it_0))=0$ for all r in $1\le r\le m_0-1$, hence (5.48) becomes

$$\begin{aligned}&-|\sum_{r=1}^{m_0-1}(x_0-y_0)^r\{\frac{(-1)^r}{r!}(\delta_0-x_0)^{m_0-r}\frac{(-1)^{m_0-r}}{(m_0-r)!}Re(f^{(m_0)}(\delta_0+it_0))\\ &\quad\pm 3\vartheta(N)\pm 2\frac{(y_0-x_0)^{m_0}}{n^2_{\alpha,\beta}}\pm\frac{1}{r!}\frac{(r+1)(\log(M))^r}{2^{2M+2}}\pm\frac{1}{r!}\vartheta(N)\}|\\ &\quad\quad+|(x_0-y_0)^{m_0}Re(\Gamma(y_0,x_0,m_0))|-\frac{|x_0-y_0|^{m_0}}{n^2_{\alpha,\beta}}<2\vartheta(N)\ .\end{aligned}\tag{5.49}$$

Therefore by the relation $x_0<\delta_0<y_0$ and m_0 being an positive even integer, which is confirmed in Definition 5.6, hence for the locally uniform inequality (5.49) it becomes

$$\begin{aligned}&-\sum_{r=1}^{m_0-1}(y_0-x_0)^r(\delta_0-x_0)^{m_0-r}|\frac{(-1)^r}{r!(m_0-r)!}Re(f^{(m_0)}(\delta_0+it_0))|\\ &\quad+(y_0-x_0)^{m_0}|Re(\Gamma(y_0,x_0,m_0))|-3\frac{(y_0-x_0)^{m_0}}{n^2_{\alpha,\beta}}<6\vartheta(N)\ .\end{aligned}\tag{5.50}$$

Further, by limit (5.33) of Theorem 5.8 which is a locally uniform convergence with perturbation $n^{-2}_{\alpha,\beta}|f^{(m_0+1)}(\delta_0+it_0)|$. Hence for $r_{x_0}:=(\delta_0-x_0)/\ (y_0-x_0)$ and for each sufficiently large M it implies the locally uniform inequality

$$\{2-\frac{1-r_{x_0}^{m_0}}{1-r_{x_0}}\}(y_0-x_0)^{m_0}|Re(f^{(m_0)}(\delta_0+it_0))|-(3+\xi)\frac{(y_0-x_0)^{m_0}}{n^2_{\alpha,\beta}}<6\vartheta(N)\ ,\tag{5.51}$$

where $\xi:=|f^{(m_0+1)}(\delta_0+it_0)|$, so

$$\{2-\frac{1-r_{x_0}^{m_0}}{1-r_{x_0}}\}|Re(f^{(m_0)}(\delta_0+it_0))|-\frac{(3+\xi)}{n^2_{\alpha,\beta}}<6\vartheta(N)(y_0-x_0)^{-m_0}\ .\tag{5.52}$$

Meanwhile if we start by considering another difference function $Re(f_M(s_{x_0,y_0}))$ at the begining, then we will eventually have another similar locally uniform inequality as that of (5.52). So here we summarize these cases which hold simultaneously true for each chosen M

$$\begin{aligned}&\{2-\frac{1-r_{x_0}^{m_0}}{1-r_{x_0}}\}|Re(f^{(m_0)}(\delta_0+it_0))|-\frac{3+\xi}{n^2_{\alpha,\beta}}<6\vartheta(N)(y_0-x_0)^{-m_0}\\ &\qquad\qquad r_{x_0}:=(\delta_0-x_0)/(y_0-x_0);\\ &\{2-\frac{1-r_{y_0}^{m_0}}{1-r_{y_0}}\}|Re(f^{(m_0)}(\delta_0+it_0))|-\frac{3+\xi}{n^2_{\alpha,\beta}}<6\vartheta(N)(y_0-x_0)^{-m_0}\\ &\qquad\qquad r_{y_0}:=(y_0-\delta_0)/(y_0-x_0).\end{aligned}\tag{5.53}$$

And by the relation $x_0<\delta_0<y_0$, for example we let $a=r_{x_0}=1/2$ then $2-(1-a^{m_0})/(1-a)=2^{-m_0+1}>0$, while the value of the function $2-(1-x^{m_0})/(1-x)$ strictly decreases from 1 to 2^{-m_0+1} on the interval $[0,1/2]$. Hence for each chosen M we then have defined the positive real numbers r_{x_0} and r_{y_0} such that $r_{x_0}+r_{y_0}=1$, so for every chosen M we always have the real number $a=r_{x_0}$ or r_{y_0} such that $0<a\le 1/2$. Hence for each of the chosen integers M then by

(5.53) and by (5.29) we always have the locally uniform inequality

$$\begin{aligned} & 2^{-m_0+1}|Re(f^{(m_0)}(\delta_0+it_0))| - \frac{3+\xi}{n_{\alpha,\beta}^2} < 6\vartheta(N)(y_0-x_0)^{-m_0} \\ & \qquad < 6\{\frac{1}{2^{N/2}} + \frac{1/2}{3^{(\log(N+1))^{\lambda_0}}}\}2^{m_0}n_{\alpha,\beta}^{2m_0}, \\ & n_{\alpha,\beta} : = M^2(\log M)^{n_0} \ , \quad n_0 := [(\log M)^2]+1. \end{aligned} \tag{5.54}$$

But by choosing the given fixed positive integer λ_0 to be $3m_0$ for the estimation (5.54), then it is impossible for each sufficiently large integer $M \geq \max\{M(K_s, m_0), N\}$ since the nonvanishing of $|Re(f^{(m_0)}(\delta_0+it_0))|$ for this choice of the local extremum δ_0+it_0 which is defined in Definition 5.6, and since the notice after Theorem 4.7 that eventually $\max\{M(K_s, m_0), N\} = N$ for each large integer N grearter than its associated minima $M(K_s, m_0) \geq n_0(N)$, hence eventually we identify M with N in the inequality (5.54). Hence we could not have the assumption of (5.27) and (5.28) for the cases $\alpha < 1-\alpha$, hence only the case $\alpha = 1-\alpha = 1/2$ survives if the assumption (5.27) is taken, and finally the proof is completed. □

Now we come to state and then prove Lemma 5.10 which is crucial in verifying the truth of Extended Riemann Hypothesis. First, similarly to the argument (2.13), then for the difference polynomial $f_N^{(r)}(s_{\alpha,\beta})$ derived from (5.3) we have the following asymptotic formula

$$\begin{aligned} f_N^{(r)}(s_{\alpha,\beta}) \asymp & \sum_{n=1}^{\infty}(\beta-\alpha)^n \sum_{k=0}^{N} \frac{(\log(k+1))^n}{n!}(-1)^k b_{k+1}(-1)^r(\log(k+1))^r \\ & \exp(-(\beta+it_0)\log(k+1)) \sum_{l=k}^{N}\binom{l}{k}\frac{1}{2^{l+1}} \ , \end{aligned} \tag{5.55}$$

which plays the role to estimate the difference polynomial $f_N^{(r)}(s_{\alpha,\beta})$ which is defined on the Riemann sub-surface $\tilde{\mathbf{S}} = \{\tilde{s_1} \in \mathbf{S} \mid e^{-1/Q_{N_0}} < |\tilde{s_1}| < 1, \tilde{s_1} = s_\sigma\}$. We note that: To estimate the right hand side of (5.55) for the case $r = 0$ is enough to prove Lemma 6.10, since the magnitude $(-1)^r(\log(k+1))^r = o((k+1)^\varepsilon)$ which makes no difference in proving any case $\phi^{(r)}(s)$ with integer r in $0 \leq r \leq m_0$ since the integer m_0 is given.

We note that: By the argument of (2.13), the right hand side polynomial is a dominant asymptotic w.r.t. the left hand side difference polynomial $f_N(s_{\alpha,\beta})$. Namely the domination is decided by comparing all the absolute values of each pair of the corresponding coefficients interpreted in (5.55) w.r.t. the same monomial term $(\beta-\alpha)^n$.

And we recall the Stirling's formula:

$$(n-1)! \asymp \sqrt{\frac{2\pi}{n}}e^{n(\log n-1)}(1+\frac{1}{12n}+O(\frac{1}{n^2})) \ , \tag{5.56}$$

by which, we will estimate the existence of a certain positive real number $n_{\alpha,\beta}$, depending on any given sufficiently large natural number N, for which it depends on the given real numbers α and β with $0 < \alpha \leq \beta < 1$.

That is, we will re-scale all of the original coefficients of the monomial term $(\beta-\alpha)^n$ by the quotients defined in (5.57) at the below, with each one's location still kept with the same monomial term $(\beta-\alpha)^n$ of the original $f_N(s_{\alpha,\beta})$ which is interpreted in (5.55).

Thus after this re-scaling, we will set all of the newly shifted coefficients with their absolute values to be less than 1. Namely we have

Lemma 5.10. *For any given real numbers α and β with $0 < \alpha < \beta < 1$, and for each integer $N \geq n_0(N)$ where for the fixed large integer $n_0(N)$ which depends on the given compact subset K_s defined in (4.6) and $f(s)$ for which it is explained at the notice after Theorem 4.7, then*

(1) *There is an integer $n_0 := [(\log N)^2] + 1$, such that for each index $n \geq n_0$, then the absolute value of each coefficient of the monomial $(\beta - \alpha)^n$ is less than 1.*

(2) *If we take the number $n_{\alpha,\beta} := N^2(\log N)^{n_0}$ to define the following newly re-scaled difference polynomial, derived from the original $f(s_{\alpha,\beta})$ interpreted in (5.3), which is denoted by*

$$f_N(s_{a,b}),$$

by introducing

$$a := \frac{\alpha}{n_{\alpha,\beta}} \quad , \quad b := \frac{\beta}{n_{\alpha,\beta}} \ . \tag{5.57}$$

Then we have for each index n in $1 \leq n \leq n_{\alpha,\beta}$, the absolute value of each coefficient of the monomial term $(\beta - \alpha)^n$ for $f_N(s_{a,b})$ is less than 1. Note: Since $n_{\alpha,\beta} > n_0$ the number formulated in the statement (1), thus after the re-scaling (5.57) it guarantees that for all indexes n, then all the absolute values of the coefficient of the monomial term $(\beta - \alpha)^n$ for $f_N(s_{a,b})$ are less than 1.

Proof. First, we note that we have the following estimation for the rear faction shown up in the formula (5.55)

$$\left| \ \frac{(-1)^k b_{k+1}(\log(k+1))^n}{n!} \sum_{l=k}^{N} \binom{l}{k} \frac{1}{2^{l+1}} \ \right| < N^{1+\epsilon} \ , \tag{5.58}$$

for all n in $0 \leq n \leq P_N$, and for all k in $0 \leq k \leq N$ and for any given $\epsilon > 0$. Now for any given large integer Q_{N_0} defined in (1.5), and by the limit defined in (5.39), then we have the following inequality for the middle faction shown up in the formula (5.55)

$$\frac{1}{(Q_{N_0})^n} \binom{P_k}{n} < \frac{(\log(k+1))^n}{n!} \ . \tag{5.59}$$

We note that for the left hand side of (5.59), it is the decisive factor for estimating the coefficient of the monomial term $(\beta - \alpha)^k$ for the difference polynomial $f_N(s_{\alpha,\beta})$ expressed in (5.55). And for its esimation, we switch it to the estimation of the right hand side of (5.59), which is the decisive factor for estimating the coefficient of the monomial term $(\beta - \alpha)^k$ for the dominant asymptotic defined in (5.55). Therefore instead of estimating the coefficient of $f_N(s_{\alpha,\beta})$ directly, we apply the Stirling's formula to the dominant right hand side of (5.55), and we start to estimate the decisive factors of the dominant asymptotic polynomial which are now on the right hand side of (5.59).

For the proof of statement (1), by (5.58) and (5.59) then the absolute value of the coefficient of the monomial $(\beta - \alpha)^n$, for the difference polynomial $f_N(s_{\alpha,\beta})$, is dominated by the number which is the one on the left hand side of the following inequality

$$\frac{N^2(\log N)^n}{n!} < \frac{1}{\sqrt{2\pi}} \exp(2\log N + n\log\log N - n\log n + n + \frac{1}{2}\log n) \ , \tag{5.60}$$

since the existence of this inequality is guaranteed by the Stirling's formula. Hence by the right hand side of the inequality (5.60), it is sufficient to require the index n satisfy the following condition

$$n\log n - n - \frac{1}{2}\log n - 2\log N - n\log\log N > 0 \ . \tag{5.61}$$

And we go forth to find out the minimum index $n = n_0$ which satisfies the above inequality.

Firstly, we want to locate the value $x = x_0$, which is greater than any given critical point of the following real-valued function

$$h(x) := x\log x - x - \frac{1}{2}\log x - 2\log N - x\log\log N \ . \tag{5.62}$$

And for this function $h(x)$, we analyze its first derivation with solution in the equation $\frac{d}{dx}h(x) = 0$.

Secondly, we define the value $x_0 := (\log N)^2$ for any given sufficiently large integer N. And then we take the number $x_0 = (\log N)^2$ to test the positivity of the function $\frac{d}{dx}h(x)$ defined over all the points $x \geq x_0$. Since we observe that for any $x \geq x_0$, then we have

$$\frac{d}{dx}h(x) \geq \frac{d}{dx}h(x_0) = 2\log\log N - \frac{1}{2(\log N)^2} - \log\log N > 0 \ . \tag{5.63}$$

Hence $\frac{d}{dx}h(x)$ is a positive function defined over all the points $x \geq x_0$, namely, $h(x)$ is an increasing function defined over all the points $x \geq x_0$.

Thirdly, for any sufficiently large N, then we have

$$h(x_0) > (\log N)^2\{2\log\log N - 1 - \log\log N\} - \log\log N - 2\log N > 0 \ . \tag{5.64}$$

Therefore by (5.63) and (5.64), then it implies that $h(x) > 0$ defined over all the points $x \geq x_0$. And this is what we want to learn from the function $h(x)$. Finally, if we pick up any index n satisfying $n \geq n_0 := [(\log(N))^2] + 1$, then it must satisfy the statement (1) which then completes the proof.

For the proof of statement (2), then the question is how to settle the real value $x = x_{\alpha,\beta}$ to satisfy the following requirement: If we re-scale all the coefficients of the monomial $(\beta-\alpha)^n$ in the original difference polynomial $f_N(s_{\alpha,\beta})$ interpreted in (5.55), first by shifting to the scale $\frac{(\beta-\alpha)^n}{x^n}$ to denote the formulation (5.57), then we want the new coefficient of the monomial $(\beta-\alpha)^n$ with its absolute value to be less than 1.

Now for the given sufficiently large integer Q_{N_0}, then by (5.58) and (5.59) we define the following re-scaling relation

$$\frac{1}{x^n}\frac{1}{(Q_{N_0})^n}\binom{P_k}{n} < \frac{1}{x^n}\frac{(\log(k+1))^n}{n!} \ , \tag{5.65}$$

and then, the original requirement in the absolute value of its coefficients to be less than 1 is switched to requiring that of the right hand side of the inequality (5.55). Hence by the right hand side of the inequality (5.55), the original requirement is switched to the requirement for the dominant asymptotic polynomial on the right hand side of (5.55) such that, the absolute value of its coefficient is to be, firstly at most

$$\frac{1}{x^n}\frac{N^2(\log N)^n}{n!} \ , \tag{5.66}$$

and then, we require the dominating number chosen on the right hand side of (5.65) be less than 1. So for our requirement, it is natural to ask the question: what are the values for the variable x such that, for all the indexes n in $1, \ldots, n_0 = [(\log N)^2] + 1$ they all satisfy the following inequality

$$\frac{1}{x^n}\frac{N^2(\log N)^n}{n!} \leq 1 \ . \tag{5.67}$$

Firstly, we start from the inequality (5.67), and then we take the following much simpler situation

$$\frac{1}{x^n}\frac{N^2(\log N)^n}{1} \leq 1 \ , \tag{5.68}$$

instead of (5.67) to evaluate the suitable values for x. So after taking the log-evaluaton on both sides of (5.68), thus we can base on the following inequality

$$\begin{aligned} n\log x \geq & 2\log N + n\log\log N \\ & \text{for any } n = 1, 2, \dots, n_0 = [(\log N)^2] + 1 \ , \end{aligned} \tag{5.69}$$

to solve the required suitable values for the variable x with which, they all meet our requirement (5.67).

Secondly, we let the range of evaluation for the variable x satisfy the following condition

$$\begin{aligned} x \geq x_{\alpha,\beta} \quad \text{where} \quad x_{\alpha,\beta} : = & N^2(\log N)^{n_0} \\ n_0 = & [(\log N)^2] + 1 \ . \end{aligned} \tag{5.70}$$

And we substitute (5.69) by a stronger condition which is stated below, namely, we require that those values for the variable x formulated in (5.69), also satisfy the following stronger condition

$$\begin{aligned} \log x \geq 2\log N + & n_0 \log\log N \\ n_0 = & [(\log N)^2] + 1 \ . \end{aligned} \tag{5.71}$$

Thirdly, we take this specific value $x = x_{\alpha.\beta}$ defined in (5.70) for our re-scaling purpose, since this specific value satisfies both the requirements (5.69) and (5.71). That is, for any index n in

$$\begin{aligned} 1 \leq n \leq n_0 < n_{\alpha,\beta} : = x_{\alpha,\beta} = & N^2(\log N)^{n_0} \\ n_0 = & [(\log N)^2] + 1 \ , \end{aligned} \tag{5.72}$$

such that, if we take the following numbers

$$a := \frac{\alpha}{n_{\alpha,\beta}} \ , \ b := \frac{\beta}{n_{\alpha,\beta}} \ , \tag{5.73}$$

thus by $n_{\alpha,\beta} > n_0$ and by statement (1), then all the absolute value of the coefficient of the monomial terms $(\beta - \alpha)^n$ for the difference polynomial $f_N(s_{a,b})$ are less than 1. ☐

6. Grand Riemann Hypothesis

For concrete examples in the automorphic functions beyond the Riemann zeta function $\zeta(s)$ and Dedekind zeta functions $\zeta_K(s)$, they are: The Dirichlet L-functions $L(s, \chi)$, the Hecke L-functions, the Artin L-functions, the Selberg class, and the Elliptic L-functions $L(E, s)$ and so on. Then in this section we will tackle the problem of Riemann hypothesis on them.

Definition 6.1. For each Dirichlet η-function $f(s)$ which is defined in (3.1) with $f(s) \neq \overline{f(s)}$ for which we define the function $\phi(s)$

$$\phi(s) := f(s)\overline{f(1-\bar{s})} \ , \quad 0 < Re(s) < 1 \ . \tag{6.1}$$

Lemma 6.2. *For each $\phi(s)$ which is defined in (6.1) with $s = \sigma + it_0$ and t_0 fixed, then for each positive integer r the r-th σ-derivative $\frac{d^r}{d\sigma^r}\phi(s)$ is*

$$\frac{d^r}{d\sigma^r}\phi(s) = \sum_{p+q=r} \binom{r}{q} f^{(p)}(s)(-1)^q\overline{f^{(q)}(1-\bar{s})} \ . \tag{6.2}$$

Proof. When we treat the complex plane $\mathbb{C}$ as the two real variables σ-t plane, then for any given complex analytic function $F(s)$ then $\frac{\partial}{\partial \bar{s}}F(s) = 0$ and $\frac{\partial}{\partial s}\overline{F(s)} = 0$. Hence by Chain-rule applying on the complex partial derivatives $\frac{\partial}{\partial \bar{s}}\phi(s)$ and $\frac{\partial}{\partial s}\overline{\phi(s)}$

$$\frac{\partial}{\partial s}\phi(s) = f^{(1)}(s)\overline{f(1-\bar{s})} \ , \ \frac{\partial}{\partial \bar{s}}\phi(s) = f(s)(-1)\overline{f^{(1)}(1-\bar{s})}$$

while in treating the real-variable σ-derivative $\frac{d}{d\sigma}\phi(s)$ with the variable t being fixed, say for $t = t_0$ and $\phi(\sigma + it_0)$ a path parametrized by real-variable σ then

$$\begin{aligned}\frac{d}{d\sigma}\phi(s) &= \frac{\partial}{\partial s}\phi(s)\frac{ds}{d\sigma} + \frac{\partial}{\partial \bar{s}}\phi(s)\frac{d\bar{s}}{d\sigma} \\ &= f^{(1)}(s)\overline{f(1-\bar{s})} + f(s)(-1)\overline{f^{(1)}(1-\bar{s})} \ ,\end{aligned}$$

with which then it implies the proof, by induction which is based on repeating r-many times of the above process. □

Definition 6.3. For each integer r in $1 \leq r \leq m_0$ since we have the similar formulation from (1.5) to (1.16) for each r-th σ-derivative $\frac{d^r}{d\sigma^r}\phi(s)$ of $\phi(s)$ which is defined in (6.1) with $0 < Re(s) < 1$, hence with each $f^{(r)}(s)$ then by Lemma 6.2 and (4.22)

$$\begin{aligned}\frac{d^r}{d\sigma^r}\phi(s) :&= \sum_{p+q=r}\binom{r}{q} f^{(p)}(s)(-1)^q\overline{f^{(q)}(1-\bar{s})} \ ; \\ f_M^{(r)}(s) :&= \sum_{n=0}^{M}\frac{1}{2^{n+1}}\sum_{k=0}^{n}\binom{n}{k}(-1)^k b_{k+1}(-1)^r(\log(k+1))^r(k+1)^{-s} \ , \\ \frac{d^r}{d\sigma^r}\phi_M(s) :&= \sum_{p+q=r}\binom{r}{q} f_M^{(p)}(s)(-1)^q\overline{f_M^{(q)}(1-\bar{s})} \ ; \\ f_M^{(r)}(s_{x_0,y_0}) :&= \tilde{f_M^{(r)}}(s_{x_0}) - \tilde{f_M^{(r)}}(s_{y_0}) \ , \text{ by definition of (4.12)} \ , \\ \tilde{f_M^{(r)}}(s_{y_0}) :&= \sum_{n=0}^{M}\frac{1}{2^{n+1}}\sum_{k=0}^{n}\binom{n}{k}(-1)^k b_{k+1}(-1)^r(\log(k+1))^r s_{y_0}^{P_k} \\ s_{y_0} :&= \exp(\frac{-(y_0+it)}{Q_{M_0}}) \ ; \\ \frac{d^r}{d\sigma^r}\phi_M(s_{x_0,y_0}) :&= \frac{d^r}{d\sigma^r}\tilde{\phi_M}(s_{x_0}) - \frac{d^r}{d\sigma^r}\tilde{\phi_M}(s_{y_0}) \ , \text{ analogously to (4.12)} \ , \\ \frac{d^r}{d\sigma^r}\tilde{\phi_M}(s_{y_0}) :&= \sum_{p+q=r}\binom{r}{q}\tilde{f_M^{(p)}}(s_{y_0})(-1)^q\overline{\tilde{f_M^{(q)}}(1-\bar{s})_{y_0}} \\ (1-\bar{s})_{y_0} :&= \exp(-(1-y_0+it_0)/Q_{M_0}) \ .\end{aligned} \tag{6.3}$$

We note that: For each given Dirichlet η-function $f(s)$ with the property $f(s) = \overline{f(s)}$ then for its defined $\phi(s)$ whose difference functions $\frac{d^r}{d\sigma^r}\phi_M(s_{x_0,y_0})$ defined in Definition 6.3, when $y_0 = 1 - x_0$, all result in as the constant function 0. Hence in the future and for the case

$f(s) = \overline{f(s)}$ it is necessarily excluded in applying the formulation of (6.3). We define

$$\mu_1(p) := \tilde{f}_M^{(p)}(s_{x_0}),\ \nu_1(q) := (-1)^q \overline{\tilde{f}_M^{(q)}((1-\bar{s})_{x_0})},$$
$$\mu_2(p) := \tilde{f}_M^{(p)}(s_{y_0}),\ \nu_2(q) := (-1)^q \overline{\tilde{f}_M^{(q)}((1-\bar{s})_{y_0})};$$

$$\begin{aligned}\frac{d^r}{d\sigma^r}\phi_M(s_{x_0,y_0}) &= \sum_{p+q=r}\binom{r}{q}\{\ \mu_1(p)\nu_1(q) - \mu_2(p)\nu_2(q)\ \} \\ &= \sum_{p+q=r}\binom{r}{q}\{\ (\mu_1(p)-\mu_2(p))\nu_1(q)\ +\ \mu_2(p)(\nu_1(q)-\nu_2(q))\ \} \\ &= \sum_{p+q=r}\binom{r}{q}\{\ f_M^{(p)}(s_{x_0,y_0})\nu_1(q)\ +\ \mu_2(p)(-1)^q\overline{f_M^{(q)}((1-\bar{s})_{x_0},(1-\bar{s})_{y_0})}\ \}.\end{aligned} \tag{6.4}$$

Similarly to Definition 5.4, then for the case $\phi(s)$ we consider:

Definition 6.4. For each integer q in $1 \le q \le m_0$, then by the notations defined in (6.4) we define the function

$$\begin{aligned}\Gamma(x_0, y_0, q) := \exp(\frac{-q\sigma_{x_0,y_0}}{Q_{M_0}}) \sum_{k=0}^{M} \frac{1}{Q_{M_0}^q}\binom{P_k}{q}(-1)^k b_{k+1} \\ \nu_1(0)\exp(\frac{-(y_0+it_0)P_k}{Q_{M_0}})\sum_{l=k}^{M}\binom{l}{k}\frac{1}{2^{l+1}} \\ + \exp(\frac{-q\sigma_{y_0,x_0}}{Q_{M_0}}) \sum_{k=0}^{M} \frac{1}{Q_{M_0}^q}\binom{P_k}{q}(-1)^k b_{k+1} \\ \mu_2(0)\exp(\frac{-(1-y_0-it_0)P_k}{Q_{M_0}})\sum_{l=k}^{M}\binom{l}{k}\frac{1}{2^{l+1}}\ ,\end{aligned} \tag{6.5}$$

such that for the polynomial $\phi_M(s_{x_0,y_0})$, its coefficient of the degree q-th term associated with the monomial $(y_0 - x_0)^q$ is $\Gamma(x_0, y_0, q)$.

$$\begin{aligned}\Omega(x_0, y_0, m_0) := \sum_{n=m_0+1}^{P_M}(y_0-x_0)^n \exp(\frac{-n\sigma_{x_0,y_0}}{Q_{M_0}}) \sum_{k=0}^{M} \frac{1}{Q_{M_0}^n}\binom{P_k}{n}(-1)^k b_{k+1} \\ \nu_1(0)\exp(\frac{-(y_0+it_0)P_k}{Q_{M_0}})\sum_{l=k}^{M}\binom{l}{k}\frac{1}{2^{l+1}}\ , \\ + \sum_{n=m_0+1}^{P_M}(y_0-x_0)^n \exp(\frac{-n\sigma_{y_0,x_0}}{Q_{M_0}}) \sum_{k=0}^{M} \frac{1}{Q_{M_0}^n}\binom{P_k}{n}(-1)^k b_{k+1} \\ \mu_2(0)\exp(\frac{-(1-y_0-it_0)P_k}{Q_{M_0}})\sum_{l=k}^{M}\binom{l}{k}\frac{1}{2^{l+1}},\end{aligned} \tag{6.6}$$

hence we have the decomposition

$$\phi_M(s_{x_0,y_0}) = \sum_{q=1}^{m_0}(y_0-x_0)^q\Gamma(x_0,y_0,q) + \Omega(x_0,y_0,m_0)\ . \tag{6.7}$$

Substituting $\phi(s)$ for $f(s)$ in proving Theorem 5.5 is based on substituting $\phi(s)$ for $f(s)$ in proving Theorem 4.7, hence we have

Theorem 6.5. *For any given integer m_0 and sufficiently large integers N and M with*

$$M \geq \max\{M(K_s, m_0), N\} \ ,$$

where for each integer $M(K_s, m_0)$ it is formulated in the proof of Theorem 4.7 and with all the integers r in $0 \leq r \leq m_0$, where $f(s)$ is the Dirichlet η-function defined in (3.1) with each derivative $f^{(r)}(s)$, then for the variable s in $0 < Re(s) < 1$ and by Definition 6.3. Then for each number

$$\vartheta(N) := \frac{1}{2^{N/2}} + \frac{1/2}{3^{(\log(N+1))^{\lambda_0}}},$$

we have the locally uniform estimations

1. *we have the similar formulations from (1.5) to (1.16) for the functions $\phi^{(r)}(s)$,*
2. *by applying the triangular inequality we have the similar asymptotics as the approximation (4.23) of Theorem 4.7 for each $\phi^{(r)}(s)$, in particular we have the finite difference function $\phi_M(s_{x_0,y_0})$ which is defined similarly to (4.12) for $f_M(s_{x_0,y_0})$,*
3. *with the basic equality $ab - cd = (a-c)b + c(b-d)$ we apply the triangular inequality to estimating the locally uniform approximation of $\phi_M(s_{x_0,y_0})$ to the difference $\phi(w_{x_0}) - \phi(w_{y_0})$, then for each $\phi_M(s_{x_0,y_0})$ we have the similar estimation as the approximation (4.24) of $f(s)$,*
4. *we have the similar estimations as those of Lemma 5.1, Lemma 5.2, Lemma 5.3 and Theorem 5.5.*

Proof. Since by (6.4) then for the functions $\mu_i(p)$, $\nu_i(q)$ whose absolute values are all bounded on the given compact subset $\{s = \sigma + it_0 \in \mathbb{C} \mid 0 < \alpha \leq \sigma \leq \beta < 1\}$ by a fixed constant C_0, and by $ab - cd = (a-c)b + c(b-d)$, and by Theorem 4.7 with applying the triangular inequality to estimating the value $|ab - cd| \leq |(a-c)b| + |c(b-d)|$, hence by Definition 6.4 and Definition 6.7, and by (4.23) of Theorem 4.7 then we have

$$(6.8)\quad |\phi^{(r)}(w_{x_0}) - \tilde{\phi^{(r)}}_M(s_{x_0})| < 2(2^{m_0}-1)C_0\vartheta(N), \quad w_{x_0} = x_0 + it_0 \ , \ s_{x_0} = \exp(\frac{-(x_0+it_0)}{Q_{M_0}}),$$

where more precisely C_0 is a constant depending on the already given data m_0, $\{f^{(r)}(w_\sigma)\}_{r=0}^{m_0}$, and K_s the given compact subset containing the points s_{x_0}, s_{y_0}, $(1-\bar{s})_{x_0}$, and $(1-\bar{s})_{y_0}$, hence the statement (2) is proved.

For Statement (3), we have the similar asymptotic as (4.24) of Theorem 4.7 for each $\phi^{(r)}(s)$. We claim that by (6.8) then for the difference function $\phi_M^{(r)}(s_{x_0,y_0})$ whose approximating $\phi^{(r)}(w_{x_0}) - \phi^{(r)}(w_{y_0})$ is in a magnitude of less than $2(2^{m_0}-1)\ C_0 2\vartheta(N)$. Since by (6.3) of Definition 6.3, $\phi_M^{(r)}(s_{x_0,y_0}) = \tilde{\phi^{(r)}}_M(s_{x_0}) - \tilde{\phi^{(r)}}_M(s_{y_0})$, and by applying twice the similar argument of (6.8) then we have

$$(6.9)\quad \begin{aligned} &|\{\phi^{(r)}(w_{x_0}) - \phi^{(r)}(w_{y_0})\} - \phi_M^{(r)}(s_{x_0,y_0})| \\ &= |\{\phi^{(r)}(w_{x_0}) - \tilde{\phi^{(r)}}_M(s_{x_0})\} - \{\phi^{(r)}(w_{y_0}) - \tilde{\phi^{(r)}}_M(s_{y_0})\}| \\ &< 2(2^{m_0}-1)C_0 2\vartheta(N) \ . \end{aligned}$$

Hence the statement (3) is proved.

For Statement (4), we proceed by applying the basic relation $ab - cd = (a-c)b + c(b-d)$ and its estimation $|ab - cd| \leq |(a-c)b| + |c(b-d)|$, then by Definition 6.4 and Definition 6.7 the analogous estimation for each $\phi^{(r)}(s)$ will follow. The reason is we substitute $\phi(s)$ for the function $f(s)$ in deriving the inequalities similar to those of Lemma 5.1, Lemma 5.2, Lemma 5.3 and Theorem 5.5. □

Now for such functions $\phi(s)$ which are defined in Definition 6.1, by $\phi(1-\bar{s}) = \overline{\phi(s)}$ we naturally conclude that: For any given pair of the complex numbers $\alpha+it_0$ and $1-\alpha+it_0$ with $0<\alpha<1$, we have either $\phi(\alpha+it_0) = \phi(1-\alpha+it_0) = 0$ or $\phi(\alpha+it_0)\phi(1-\alpha+it_0) \neq 0$.

Therefore we come to assume that for any such complex point $\alpha+it_0$ which is one of the zeros of the given Dirichlet η-function $f(s)$ and we formulate it as:

$$\phi(\alpha+it_0) = \phi(1-\alpha+it_0) = 0 \ , \quad \text{with } 0<\alpha\leq 1/2 \ . \tag{6.10}$$

While for $\phi(\alpha+it_0)=0$, then by $\phi(1-\bar{s}) = \overline{\phi(s)}$ we have $\phi(1-\alpha+it_0)=0$ with the same order of zero at both the points $\alpha+it_0$ and $1-\alpha+it_0$.

Then we apply the Mean Valued Theorem and the Intermediate Valued Theorem to the real-valued functions $Re(\phi(s))$ and $Im(\phi(s))$, so that we are able to locate the positions of the two positive real numbers x_0 and y_0 such that, say for $Re(\phi(s))$

$$Re(\phi(x_0+it_0)) = Re(\phi(y_0+it_0)) \ , \quad \text{with } 0<\alpha\leq x_0\leq y_0\leq 1-\alpha<1 \ , \tag{6.11}$$

with the real value $(y_0-x_0)>0$ as small as we will.

We let δ_0+it_0 be the unique local extremum, between the chosen x_0+it_0 and y_0+it_0 which are defined in (6.11) with the real value $(y_0-x_0)>0$ as small as we will, for which it is set for either the real-valued function $Re(\phi(s))$ or the real-valued function $Im(\phi(s))$ on the line segment $L := \{s=\sigma+it_0\in\mathbb{C} \mid \alpha\leq\sigma\leq 1-\alpha\}$, where $\phi(s) = Re(\phi(s)) + iIm(\phi(s))$. By the assumption (6.10) then for $Re(\phi(s))$ with Lemma 6.2:

Definition 6.6. For the given positive integer m_0, we define it to be the index of the first nonvanishing coefficient $\frac{1}{m_0!}Re(\phi^{(m_0)}(\delta_0+it_0))$ of the Talyor series expansion of $Re(\phi(s))$ in $s\in L$ with respect to the point δ_0+it_0, for which it is induced from the Taylor expansion of $\phi(s)$ defined on the complex plane $\mathbb{C}$. Therefore for those real numbers y_0 which are defined in (6.11), the first nonvanishing coefficients of the Taylor series $Re(\phi(y_0+it_0)) - Re(\phi(\delta_0+it_0))$ is the real number $\frac{1}{m_0!}Re(\phi^{(m_0)}(\delta_0+it_0))$ and m_0 is an even positive integer, since δ_0+it_0 is the chosen local extremum.

Moreover, for the number $n_{\alpha,\beta} := M^2(\log M)^{n_0}$, and the integer $n_0 := [(\log M)^2]+1$ which are defined in Lemma 6.10 at the below, we restrict the chosen real numbers x_0 and y_0, defined in (6.11), to satisfy

$$\frac{1}{2}(\frac{1}{n_{\alpha,\beta}})^2 < (y_0-x_0) < (\frac{1}{n_{\alpha,\beta}})^2 \ , \quad x_0<\delta_0<y_0 \ . \tag{6.12}$$

Similarly to Definition 5.7, then for the case $\phi^{(r)}(s)$ with $r\geq 1$ we consider:

Definition 6.7. For each integer v in $1\leq v\leq m_0-r$, r in $1\leq r\leq m_0-1$ we define

$$\begin{aligned} \Gamma'(x_0,y_0,v) : &= \sum_{p+q=v}\binom{r}{q}\{\ \Gamma''(x_0,y_0,p)\nu_1(q) + \mu_2(p)\Gamma'''(1-x_0,1-y_0,q)\ \}, \\ \nu_1(q) : &= (-1)^q\overline{\tilde{f_M^{(q)}}((1-\bar{s})_{x_0})},\ \mu_2(p) := \tilde{f_M^{(p)}}(s_{y_0})(-1)^q, \end{aligned} \tag{6.13}$$

where $\Gamma''(x_0,y_0,p)$, $\Gamma'''(1-x_0,1-y_0,q)$ are the functions similar to the function defined in (6.5), such that for the polynomial $\phi_M^{(r)}(s_{x_0,y_0})$, its coefficient of the degree v-th term associated with the monomial $(y_0-x_0)^v$ is $\Gamma'(x_0,y_0,v)$.

$$\begin{aligned} \Omega'(x_0,y_0,m_0-r) : &= \sum_{p+q=v}\binom{r}{q}\{\ \Omega''(x_0,y_0,m_0-r)\nu_1(q) + \mu_2(p)\Omega'''(x_0,y_0,m_0-r)\ \}, \\ \nu_1(q) : &= (-1)^q\overline{\tilde{f_M^{(q)}}((1-\bar{s})_{x_0})},\ \mu_2(p) := \tilde{f_M^{(p)}}(s_{y_0})(-1)^q, \end{aligned} \tag{6.14}$$

where $\Omega''(x_0, y_0, m_0 - r)$, $\Omega'''(x_0, y_0, m_0 - r)$ are the functions similar to the function defined in (6.6), hence we have the decomposition

$$\phi_M^{(r)}(s_{x_0,y_0}) = \sum_{v=1}^{m_0-r} (y_0 - x_0)^v \Gamma'(x_0, y_0, v) + \Omega'(x_0, y_0, m_0 - r) \ . \tag{6.15}$$

Substituting $\phi(s)$ for $f(s)$ in proving Theorem 5.8 is based on substituting $\phi(s)$ for $f(s)$ in proving Theorem 5.5 for which it is based on Theorem 6.5, hence we have the analogue:

Theorem 6.8. *For the function $\Gamma(x_0, y_0, m_0)$ which is defined in Definition 6.4 with each integer $M \geq \max\{M(K_s, m_0), N\}$ for which we assume the conditions of Lemma 5.1 and Lemma 5.3, and by Definition 6.6 and (6.12). Then for each number*

$$\vartheta(N) := \frac{1}{2^{N/2}} + \frac{1/2}{3^{(\log(N+1))^{\lambda_0}}},$$

we have the locally uniform estimations

(1) *The locally uniform convergence limit on K_s with the perturbation $C_1\{n_{\alpha,\beta}^{-2}|\phi^{(m_0+1)}(\delta_0 + it_0)| + 2\vartheta(N)\}$ where $C_1 := 2(2^{m_0} - 1)$*

$$\lim_{\substack{|\log(k+1) - \frac{P_k}{Q_{M_0}}| < \frac{1}{M^2 2^{2M+2}} \\ 0 \leq k \leq M \\ N\to\infty,\ M\to\infty}} Re(\Gamma(x_0, y_0, m_0)) = \frac{(-1)^{m_0}}{m_0!} Re(\phi^{(m_0)}(\delta_0 + it_0)) \neq 0 \ . \tag{6.16}$$

(2) *The following locally uniform estimation for the functions $\Omega(x_0, y_0, u)$ with integers u in $1 \leq u \leq m_0$, which are defined in Definition 6.4*

$$| \ \Omega(x_0, y_0, u) \ | < C_1 \frac{(y_0 - x_0)^{m_0}}{n_{\alpha,\beta}^2} \ , \tag{6.17}$$

where for the number $n_{\alpha,\beta} := M^2(\log M)^{n_0}$, and the integer $n_0 := [(\log M)^2] + 1$ which are defined in Lemma 6.10 at the below.

(3) *For each integer r in $1 \leq r \leq m_0 - 1$, exactly the same locally uniform inequalities of (6.17) which also work with $|\Omega'(x_0, y_0, m_0 - r)|$, where $\Omega'(x_0, y_0, m_0 - r)$ and $\Omega(x_0, y_0, m_0 - r)$ are defined in Definition 6.7 and Definition 6.4 respectively.*

(4) *By (5.23) of Theorem 5.5 and (6.17), then for each integer r in $1 \leq r \leq m_0 - 1$ we have the following locally uniform estimation*

$$\begin{aligned} &|Re(\phi^{(r)}(x_0 + it_0)) - (\delta_0 - x_0)^{m_0-r} \frac{(-1)^{m_0-r}}{(m_0 - r)!} Re(\phi^{(m_0)}(\delta_0 + it_0))| \\ &\qquad\qquad < C_1\{3\vartheta(N) + 2\frac{(y_0 - x_0)^{m_0}}{n_{\alpha,\beta}^2}\}. \end{aligned} \tag{6.18}$$

Substituting $\phi(s)$ for $f(s)$ in proving Theorem 5.9 is based on Theorem 6.8, hence we have the analogous extension of Theorem 5.9:

Theorem 6.9 (Grand Riemann Hypothesis)**.** *For all of the Dirichlet η-functions $f(s)$ and all of their associated Dirichlet series $D(s)$ which are defined in (3.1), while assuming the truth of Lemma 6.10. Then for $f(s)$ and $f^{(k)}(s)$ the k-th derivative of $f(s)$, and for $D(s)$ and $D^{(k)}(s)$ the k-th derivative of $D(s)$, all of their nontrivial zeros are contained in the vertical line $Re(s) = 1/2$.*

Proof. We confer the reader first to review the argument from (5.45) to (5.54) in proving Theorem 5.9, while for the last and decisive locally uniform inequality (5.54) we repeat it in $2(2^{m_0}-1)$-many times such that, for the new (5.54) which we want it is the following

$$2^{-m_0+1}|Re(\phi^{(m_0)}(\delta_0+it_0))| - 2(2^{m_0}-1)\frac{3+\xi}{n_{\alpha,\beta}^2} < 2(2^{m_0}-1)6\vartheta(N)(y_0-x_0)^{-m_0}\ . \tag{6.19}$$

□

Now we come to state and then prove Lemma 6.10 which is crucial in verifying the truth of Grand Riemann Hypothesis. First, similarly to the argument (2.13), now for the difference polynomial $\phi_N^{(r)}(s_{\alpha,\beta})$ defined in (6.3) then by Definition 5.7 it implies the asymptotic formula

$$\phi_N^{(r)}(s_{\alpha,\beta}) = \sum_{p+q=r}\binom{r}{q}\{\ f_M^{(p)}(s_{\alpha,\beta})\nu_1(q)\ +\ \mu_2(p)(-1)^q\overline{f_M^{(q)}((1-\bar{s})_\alpha,(1-\bar{s})_\beta)}\ \},$$

$$f_N^{(p)}(s_{\alpha,\beta}) \asymp \sum_{n=1}^{\infty}(\beta-\alpha)^n\sum_{k=0}^{N}\frac{(\log(k+1))^n}{n!}(-1)^k b_{k+1}(-1)^p(\log(k+1))^p \tag{6.20}$$

$$\exp(-(\beta+it_0)\log(k+1))\sum_{l=k}^{N}\binom{l}{k}\frac{1}{2^{l+1}},$$

for which we place it to estimate the difference polynomial $\phi_N^{(r)}(s_{\alpha,\beta})$ which is defined on the Riemann sub-surface $\tilde{\mathbf{S}} = \{\tilde{s_1}\in\mathbf{S} \mid e^{-1/Q_{N_0}} < |\tilde{s_1}| < 1, \tilde{s_1} = s_\sigma\}$. We note that: To estimate the right hand side of (6.20) for the case $r=0$ is enough to prove Lemma 6.10, since the magnitude $|\mu_1(q)|+|\nu_2(p)| < 2C_0$ the constant defined in (6.8) and $(\log(k+1))^{m_0} = o((k+1)^\varepsilon)$, thus it makes no difference in proving each case $\phi^{(r)}(s)$ with integer r in $0\le r\le m_0$ since the integer m_0 is given.

We note that: By the argument of (2.13), the right hand side polynomial is a dominant asymptotic w.r.t. the left hand side difference polynomial $\phi_N^{(r)}(s_{\alpha,\beta})$. Namely the domination is decided by comparing all the absolute values of each pair of the corresponding coefficients interpreted in (6.20) w.r.t. the same monomial term $(\beta-\alpha)^n$. While in estimating the magnitude of its coefficients, we carry it out by the above interpretation for the asymptotic (6.20), such that we proceed by applying the basic relation $ab-cd=(a-c)b+c(b-d)$, and then in the same procedure of proving Lemma 5.10 with applying the triangular inequality to estimate the value $|ab-cd|\le|(a-c)b|+|c(b-d)|$. Hence we have

Lemma 6.10. *For the function $\phi(s)$ defined in (6.1), and for any given integer $M\ge N$ where for the sufficiently large ingter N which is given by the same condition in Lemma 5.10. Then for all the integers r in $0\le r\le m_0$ and all the difference functions $\phi_M^{(r)}(s_{x_0,y_0})$, they all have the same numerical data in the number $n_{\alpha,\beta} := M^2(\log M)^{n_0}$ and the integer $n_0 := [(\log M)^2]+1$ which are the same as those for Lemma 5.10.*

By Theorem 6.9, hence we have solved the problem of the non-existence for the Siegel's zeros. Namely

Theorem 6.11 (Siegel's zeros)**.** *On the critical strip $0<Re(s)<1$ and for each of the Dirichlet series $D(s)$ which is defined in (3.1), then*

1. *it is free of having any Siegel's zero inside the critical strip $0<Re(s)<1$,*
2. *for the $\gamma(s)$-factor of such Dirichlet seiries $D(s)$, it always evaluates with the non-zero finite complex value on the critical strip $0<Re(s)<1$.*

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