The Distribution Of Prime Numbers And The Continued Fraction

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Abstract. In this paper, we discovered a new sequence contains only ones and the prime numbers, wich can be calculated in two different ways that give the same result, the first way using the greatest common divisor (gcd), the second way consisting of using the denominator of the continued fraction defined by

$$\frac{B_m(n)}{A_m(n)} = \cfrac{1}{2 - \cfrac{3}{3 - \cfrac{4}{4 - \cfrac{5}{\ddots (n-1) - \cfrac{n}{m}}}}}$$

Our sequence defined by

$$a_m(n) = \frac{|A_m(n)|}{gcd\big(A_m(n), B_m(n)\big)}$$

Where $|x|$ denotes the absolute value of x.

Introduction

A continued fraction is an expression of the form

$$a_0 + \cfrac{b_0}{a_1 + \cfrac{b_1}{a_2 + \cfrac{b_2}{\ddots}}}$$

Other notation

$$a_0 + \frac{b_0}{a_1 +} \frac{b_1}{a_2 +} \frac{b_2}{a_3 +} \cdots$$

Where a_i and b_i are either rational numbers or real numbers.

The distribution of prime numbers has been analyzed for a formula helpful in generating the prime numbers or testing if the given numbers is prime. In this paper, we present some known formulas.

Mills showed that there exists a real number $A > 1$ such that $f(n) = \left[\, A^{3^n} \,\right]$ is a prime number for any integers n, approximately A=1.306377883863,.. (see A051021). The first few values

$f(n) =$ {2, 11, 1361, 2521008887, 16022236204009818131831320183,..}, (see A051254)

Euler's quadratic polynomial $n^2 + n + 41$ is prime for all n between 0 and 39, however, it is not prime for all integers.

The Rowland sequence of prime numbers composed entirely of 1's and primes, the sequence defined by the recurrence relation

$$r(n) = r(n-1) + \gcd(n, r(n-1));\;\; r(1) = 7$$

The sequence of differences $r(n+1) - r(n)$

1, 1, 1, 5, 3, 1, 1, 1, 1, 11, 3, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 23, 3, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1,1, 1, 1, 47, 3, 1, 5, 3, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 101, 3, 1, 1, 7, 1, 1, 1, 1, 11, 3, 1, 1, 1, 1, 1, 13, 1, 1, 1, 1, 1, 1, 1, 1, 1,.. (see A132199).

For more details and formulas see [1] and [2]. In this paper, we present an interesting sequence which plays the same role as Rowland's sequence composed of a prime number or 1. Moreover, our sequence gives all distinct prime numbers in order (conjecture 2 except the prime numbers 2 and 3).

In this paper, we use the recursive formula defined by

$$b(n) = (n+2)\big(b(n-1) - b(n-2)\big)$$

With the starting conditions $b(-1) = 0$ and $b(0) = 1$.

Conjecture 1. The continued fraction

$$\frac{b(n-2) + b(n-3)}{2n-1} = \cfrac{1}{2 - \cfrac{3}{3 - \cfrac{4}{4 - \cfrac{5}{\ddots\; (n-1) - \cfrac{n}{n+1}}}}};\, n \geq 3$$

The following expression finds all odd prime numbers

$$a(n) = \frac{2n-1}{gcd\big(2n-1, b(n-2) + b(n-3)\big)};\, n \geq 2$$

Where gcd(x, y) denotes the greatest common divisor of x and y.

The values of $a(n)$

3, 5, 7, 3, 11, 13, 1, 17, 19, 1, 23, 1, 1, 29, 31, 1, 1, 37, 1, 41, 43, 1, 47, 1, 1, 53, 1, 1, 59, 61, 1, 1, 67, 1, 71, 73, 1, 1, 79, 1, 83, 1, 1, 89, 1, 1, 1, 97, 1, 101, 103, 1, 107, 109, 1, 113, 1, 1, 1, 1, 1, 1, 127, 1, 131, 1, 1, 137, 139, 1, 1, 1, 1, 1, 149, 151, 1, 1, 157, 1, 1, 163, 1, 167,…

For $n \geq 2$, $a(n) = 2n-1$ if $2n-1$ is prime (except for n=5), 1 otherwise .

Conjecture 2. The continued fraction

$$\frac{3b(n-3) - b(n-2)}{2n-3} = \cfrac{1}{2 - \cfrac{3}{3 - \cfrac{4}{4 - \cfrac{5}{\ddots\; (n-1) - \cfrac{n}{n-3}}}}};\, n \geq 4$$

The following expression finds all odd prime numbers of the form $2n-3$ (except for the prime 3) in order.

$$a(n) = \frac{2n-3}{gcd\left(2n-3, 3b(n-3)-b(n-2)\right)}; n \geq 2$$

The values of $a(n)$

1, 1, 5, 7, 1, 11, 13, 1, 17, 19, 1, 23, 1, 1, 29, 31, 1, 1, 37, 1, 41, 43, 1, 47, 1, 1, 53, 1, 1, 59, 61, 1, 1, 67, 1, 71, 73, 1, 1, 79, 1, 83, 1, 1, 89, 1, 1, 1, 97, 1, 101, 103, 1, 107, 109, 1, 113, 1, 1, 1, 1, 1, 1, 127, 1, 131, 1, 1, 137, 139, 1, 1, 1, 1, 149, 151, 1, 1, 157, 1, 1, 163, 1, 167,…

For $n \geq 4$, $a(n) = 2n-3$ if $2n-3$ is prime, 1 otherwise .

Conjecture 3. The continued fraction

$$\frac{b(n-3)+nb(n-4)}{n^2-n-1} = \cfrac{1}{2-\cfrac{3}{3-\cfrac{4}{4-\cfrac{5}{\ddots\,(n-1)-\cfrac{n}{n-(n+1)}}}}}; n \geq 3$$

The following expression finds all odd prime numbers which ends with a 1 or 9 .

$$a(n) = \frac{n^2-n-1}{gcd(n^2-n-1,\ b(n-3)+nb(n-4))}\ ;\ for\ n \geq 2$$

The values of $a(n)$

1, 5, 11, 19, 29, 41, 11, 71, 89, 109, 131, 31, 181, 19, 239, 271, 61, 31, 379, 419, 461, 101, 29, 599, 59, 701, 151, 811, 79, 929, 991, 211, 59, 41, 1259, 1, 281, 1481, 1559, 149, 1721, 1, 61, 1979, 2069, 2161, 1, 2351, 79, 2549, 241, 1, 2861, 2969, 3079, 3191,…(see A356247)

We conjectured that :

∗ Every term of this sequence is either a prime number or 1.

∗ Except for 5, the primes all appear exactly twice, such that

$$a(n) = a(a(n)-n+1)$$

Consequently, let us consider the values of n and m such that we get:

$$a(n) = a(m) = n+m-1$$

And

$$a(n) = a(m) = \gcd(n^2-n-1, m^2-m-1)$$

Conjecture 4. The continued fraction

$$\frac{2b(n-3)+nb(n-4)}{n^2-2} = \cfrac{1}{2-\cfrac{3}{3-\cfrac{4}{4-\cfrac{5}{\ddots\,(n-1)-\cfrac{n}{n-(n+2)}}}}};\ for\ n \geq 3$$

The expression of the sequence $a(n)$ is as follows

$$a(n) = \frac{n^2 - 2}{gcd\big(n^2 - 2,\ 2b(n-3) + nb(n-4)\big)} \ \ ; for\ n \geq 3$$

The values of $a(n)$.

7, 7, 23, 17, 47, 31, 79, 7, 17, 71, 167, 97, 223, 127, 41, 23, 359, 199, 439, 241, 31, 41, 89, 337, 727, 1, 839, 449, 137, 73, 1087, 577, 1223, 647, 1367, 103, 1, 47, 73, 881, 1, 967, 1, 151, 2207, 1151, 2399, 1249, 113, 193, 401, 1, 3023, 1567, 191, 41, 71…

The sequence $a(n)$ takes only 1's and primes.

Conjecture 5. The continued fraction

$$\frac{(n+1)b(n-3) - b(n-4) - (n-1)b(n-5)}{3n-2} = \cfrac{1}{2 - \cfrac{3}{3 - \cfrac{4}{4 - \cfrac{5}{\ddots\, (n-1) - \cfrac{n}{n+2}}}}}$$

The expression of the sequence $a(n)$ is as follows

$$a(n) = \frac{3n-2}{gcd\big(3n-2,\ (n+1)b(n-3) - b(n-4) - (n-1)b(n-5)\big)} \ \ ; for\ n \geq 3$$

The values of $a(n)$.

7 , 5, 13, 2, 19, 11, 5, 1, 31, 17, 37, , 1, 43, 23, 1, 1, 1, 29, 61, 1, 67, 1, 73, 1, 79, 41, 1, 1, 1, 47, 97, 1, 103, 53, 109, 1, 1, 59, 1, 1, 127, 1, 1, 1, 139, 71, 1, 1, 151, 1, 157, 1, 163, 83, 1, 1, 1, 89, 181, 1, 1, 1, 193, 1, 199, 101, 1, 1, 211,…

The sequence $a(n)$ contains only ones and the primes.

Conjecture 6. The continued fraction

$$\frac{(n+2)b(n-3) - b(n-4) - (n-1)b(n-5)}{4n-3} = \cfrac{1}{2 - \cfrac{3}{3 - \cfrac{4}{4 - \cfrac{5}{\ddots\, (n-1) - \cfrac{n}{n+3}}}}}$$

The expression of the sequence $a(n)$ is as follows

$$a(n) = \frac{4n-3}{gcd\big(4n-3,\ (n+2)b(n-3) - b(n-4) - (n-1)b(n-5)\big)} \ \ ; for\ n \geq 3$$

The values of $a(n)$.

3, 13, 17, 7, 5, 29, 11, 37, 41, 1, 7, 53, 19, 61, 1, 23, 73, 1, 1, 1, 89, 31, 97, 101, 1, 109, 113, 1, 1, 1, 43, 1, 137, 47, 1, 149, 1, 157, 1, 1, 1, 173, 59, 181, 1, 1, 193, 197, 67, 1, 1, 71, 1, 1, 1, 229, 233, 79, 241, 1, 83, 1, 257, 1, 1, 269, 1, 277,…

The sequence $a(n)$ takes only 1's and primes.

Generalisation

The unreduced denominator $a_m(n)$ can be calculated by using the denominator of the continued fraction as follows

$$\frac{mb(n-3)-nb(n-4)}{n(m-n+2)-m}=\cfrac{1}{2-\cfrac{3}{3-\cfrac{4}{4-\cfrac{5}{\ddots\,(n-1)-\cfrac{n}{m}}}}}$$

Where n is a positive integers and m is a real number.

we can obtain the sequence of the unreduced denominator of the continued fraction as follows

$$a_m(n)=\frac{|n(m-n+2)-m|}{gcd\big(n(m-n+2)-m, mb(n-3)-nb(n-4)\big)}$$

Remark

For $m=n+1$, we obtain the sequence in the conjecture 1.

For $m=n-3$, we find the sequence in the conjecture 2.

For $m=-1$, we find the sequence in the conjecture 3.

For $m=-2$, we obtain the sequence in the conjecture 4.

For $m=n+2$, we obtain the sequence in the conjecture 5.

For $m=n+3$, we obtain the sequence in the conjecture 6.

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References

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