AN APPROXIMATION FOR $\frac{\sqrt{2}}{2}$ USING TRIANGULAR NUMBERS

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Abstract. In this short paper, I prove that the difference between the square roots of two consecutive triangular numbers tends to $\frac{\sqrt{2}}{2}$, the reciprocal of $\sqrt{2}$, as the triangular numbers tend to infinity. I believe this relationship between $\sqrt{2}$ and triangular numbers is previously unknown.

The sequence of triangular numbers, $T_{(n)}$, where $T_{(1)} = 1$, begins:

1, 3, 6, 10, 15, 21, 28, 36, 45, 55, 66, 78, 91, 105, 120, 136, 153, 171, 190, 210, 231, 253, 276, 300, 325, 351, 378, 406, 435, ... (sequence A000217 in the OEIS).

The formula is given as:

$$T_{(n)} = \sum_{k=1}^{n} k = 1 + 2 + 3 + 4 + ... + n = \frac{n(n+1)}{2}. \tag{0.1}$$

As n tends to infinity, the difference between the square roots of two consecutive triangular numbers tends to $\frac{\sqrt{2}}{2}$ (= 0.707106781 to 9 decimal places), as given in the following formula:

$$\lim_{n\to\infty} \sqrt{T_{(n+1)}} - \sqrt{T_{(n)}} = \frac{\sqrt{2}}{2}. \tag{0.2}$$

When n is small, as in the case of $n = 7$, it follows that:

$$\sqrt{T_{(8)}} - \sqrt{T_{(7)}} = \sqrt{36} - \sqrt{28} = 0.708926927 \approx \frac{\sqrt{2}}{2}.$$

This is correct only to 2 decimal places. However, when $n = 500$, say, the result is correct to 5 decimal places:

$$\sqrt{T_{(501)}} - \sqrt{T_{(500)}} = \sqrt{125751} - \sqrt{125250} = 0.707107135 \approx \frac{\sqrt{2}}{2}.$$

Proof.

Prove that

$$\lim_{n\to\infty} \sqrt{T_{(n+1)}} - \sqrt{T_{(n)}} = \frac{\sqrt{2}}{2}.$$

Using the general formula for triangular numbers in (0.1), it follows that:

$$\sqrt{\frac{(n+1)(n+2)}{2}} - \sqrt{\frac{n(n+1)}{2}} = \frac{\sqrt{2}}{2}.$$

$$\Rightarrow \sqrt{(n+1)(n+2)} - \sqrt{n(n+1)} = 1.$$

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But as n tends to infinity, this becomes

$$\sqrt{(n+1)^2} - \sqrt{n^2} = 1$$
$$\Rightarrow (n+1) - n = 1.$$

$\square$

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