THE NOTION OF OLLOIDS AND THE ERDŐS-MOSER EQUATION

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Abstract. We introduce and develop the notion of the **olloid**. We apply this notion to study the Erdős-Moser equation.

1. **Introduction**

The Erdős-Moser equation is an equation of the form

$$1^k + 2^k + \cdots + m^k = (m+1)^k$$

where m and k are positive integers. The only known solution to the equation is $1^1 + 2^1 = 3^1$ and Paul Erdős is known to have conjectured that the equation has no further solution. The exponent k and the arguments in the the Erdős-Moser equation has also be studied quite extensively. In other words, several contraints on the exponent k and the argument m of the Erdős-Moser equation have been studied under a presumption that other solutions - if any -exists. In particular, it has been shown that k must be divisible by 2 and that there is no solution with $m < 10^{1000000}$ [1]. The methods introduced by Moser were later refined and adapted to show that $m > 1.485 \times 10^{9321155}$ [2]. This was improved to the lower bound $m > 2.7139 \times 10^{1,667,658,416}$ in [5] via large scale computation of $\ln(2)$. It is also shown (see [3]) that $6 \leq k+2 < m < 2k$. It is also shown that $lcm(1, 2, \cdots, 200)$ must divide k and that any prime factor of $m+1$ must be irregular and > 1000 [4]. In 2002, it was shown that all primes $200 < p < 1000$ must divide the exponent k in the Erdős-Moser equation

$$1^k + 2^k + \cdots + m^k = (m+1)^k$$

In this paper we introduce and study the notion of the **olloid** and develop a technique for extending the solution of the Erdős-Moser equation upto exponents k. In particular, we obtain the following result

Theorem 1.1 (The extension method)**.** *If the equation*

$$\sum_{n=1}^{s} n^k = (s+1)^k$$

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has a solution and there exist some $r \in \mathbb{N}$ such that

$$1 - \frac{1}{g(s)^r} > \int_1^s \frac{g'(t)}{g(t)^2} dt + \frac{1}{g(s)} \int_1^s \frac{g'(t)}{g(t)^2} dt + \cdots + \frac{1}{g(s)^{r-1}} \int_1^s \frac{g'(t)}{g(t)^2} dt$$

with

$$g(i) := \frac{s+1-i}{(s+1)}$$

for $1 \leq i \leq s$. Then the equation

$$\sum_{n=1}^{s} n^{k+r} = (s+1)^{k+r}$$

also has a solution.

This result is consequence of the more fundamental results using the notion of the **olloid**

Lemma 1.2 (Expansion principle)**.** *Let $\mathbb{F}_s^k$ be an s-dimensional **olloid** of degree k for a fixed $k \in \mathbb{N}$. If $g : \mathbb{N} \longrightarrow \mathbb{R}^+$ is a generator with continuous derivative on $[1, s]$ and decreasing on $\mathbb{R}^+$ such that*

$$1 - \frac{1}{g(s)^r} > \int_1^s \frac{g'(t)}{g(t)^2} dt + \frac{1}{g(s)} \int_1^s \frac{g'(t)}{g(t)^2} dt + \cdots + \frac{1}{g(s)^{r-1}} \int_1^s \frac{g'(t)}{g(t)^2} dt$$

*for $r \in \mathbb{N}$ then $g : \mathbb{N} \longrightarrow \mathbb{R}^+$ is also a generator of the **olloid** $\mathbb{F}_s^{k+r}$ of degree $k + r$.*

2. **The notion of the olloid**

In this section we launch the notion of the **olloid** and prove a fundamental lemma, which will be relevant for our studies in the sequel.

Definition 2.1. Let $\mathbb{F}_s^k := \left\{(u_1, u_2, \ldots, u_s) \in \mathbb{R}^s \mid \sum\limits_{i=1}^{s} u_i^k = 1\right\}$. Then we call $\mathbb{F}_s^k$ an s-dimensional **olloid** of degree k. We say $g : \mathbb{N} \longrightarrow \mathbb{R}$ is a generator of the s-dimensional olloid of degree k if there exists some vector $(v_1, v_2, \ldots, v_s) \in \mathbb{F}_s^k$ such that $v_i = g(i)$ for each $1 \leq i \leq s$.

Question 2.2. Does there exists a fixed generator $g : \mathbb{N} \longrightarrow \mathbb{R}$ with infinitely many olloids?

Remark 2.3. While it may be difficult to provide a general answer to question 2.2, we can in fact provide an answer by imposition certain conditions for which the generator of the **olloid** must satisfy. In particular, we launch a basic and a fundamental principle relevant for our studies in the sequel.

Lemma 2.4 (Expansion principle)**.** *Let $\mathbb{F}_s^k$ be an s-dimensional **olloid** of degree k for a fixed $k \in \mathbb{N}$. If $g : \mathbb{N} \longrightarrow \mathbb{R}^+$ is a generator with continuous derivative on $[1, s]$ and decreasing on $\mathbb{R}^+$ such that*

$$1 - \frac{1}{g(s)^r} > \int_1^s \frac{g'(t)}{g(t)^2} dt + \frac{1}{g(s)} \int_1^s \frac{g'(t)}{g(t)^2} dt + \cdots + \frac{1}{g(s)^{r-1}} \int_1^s \frac{g'(t)}{g(t)^2} dt$$

*for $r \in \mathbb{N}$ then $g : \mathbb{N} \longrightarrow \mathbb{R}^+$ is also a generator of the **olloid** $\mathbb{F}_s^{k+r}$ of degree $k + r$.*

Proof. Suppose $g : \mathbb{N} \longrightarrow \mathbb{R}^{+}$ is a generator of the **olloid** $\mathbb{F}_s^k$ with continuous derivative on $[1, s]$. Then there exists a vector $(v_1, v_2, \ldots, v_s) \in \mathbb{F}_s^k$ such that $v_i = g(i)$ for each $1 \leq i \leq s$, so that we can write

$$\sum_{i=1}^{s} g(i)^k = 1.$$

Let us assume to the contrary that there exists no $r \in \mathbb{N}$ such that $g : \mathbb{N} \longrightarrow \mathbb{R}^{+}$ is a generator of the **olloid** $\mathbb{F}_s^{k+r}$. By applying the summation by parts, we obtain the inequality

$$\frac{1}{g(s)} \sum_{i=1}^{s} g(i)^{k+1} \geq 1 - \int_{1}^{s} \frac{g'(t)}{g(t)^2} dt \tag{2.1}$$

by using the inequality

$$\sum_{i=1}^{s} g(i)^{k+1} < \sum_{i=1}^{s} g(i)^k = 1.$$

By applying summation by parts on the left side of (2.1) and using the contrary assumption, we obtain further the inequality

$$\frac{1}{g(s)^2} \sum_{i=1}^{s} g(i)^{k+2} \geq 1 - \int_{1}^{s} \frac{g'(t)}{g(t)^2} dt - \frac{1}{g(s)} \int_{1}^{s} \frac{g'(t)}{g(t)^2} dt. \tag{2.2}$$

By induction we can write the inequality as

$$\frac{1}{g(s)^r} \sum_{i=1}^{s} g(i)^{k+r} \geq 1 - \int_{1}^{s} \frac{g'(t)}{g(t)^2} dt - \frac{1}{g(s)} \int_{1}^{s} \frac{g'(t)}{g(t)^2} dt - \cdots - \frac{1}{g(s)^{r-1}} \int_{1}^{s} \frac{g'(t)}{g(t)^2} dt$$

for any $r \geq 2$ with $r \in \mathbb{N}$. Since $g : \mathbb{N} \longrightarrow \mathbb{R}^{+}$ is decreasing, it follows that

$$1 - \int_{1}^{s} \frac{g'(t)}{g(t)^2} dt - \frac{1}{g(s)} \int_{1}^{s} \frac{g'(t)}{g(t)^2} dt - \cdots - \frac{1}{g(s)^{r-1}} \int_{1}^{s} \frac{g'(t)}{g(t)^2} dt > 1$$

and using the requirement

$$1 - \frac{1}{g(s)^r} > \int_{1}^{s} \frac{g'(t)}{g(t)^2} dt + \frac{1}{g(s)} \int_{1}^{s} \frac{g'(t)}{g(t)^2} dt + \cdots + \frac{1}{g(s)^{r-1}} \int_{1}^{s} \frac{g'(t)}{g(t)^2} dt$$

for $r \in \mathbb{N}$, we have the inequality

$$\begin{aligned} 1 &= \sum_{i=1}^{s} g(i)^k \\ &\geq \sum_{i=1}^{s} g(i)^{k+r} > 1 \end{aligned}$$

which is absurd. This completes the proof of the Lemma. □

Lemma 2.4 - albeit fundamental - is ultimately useful for our study of the Erdős-Moser equation. It can be seen as a tool for extending the solution of equations of the form

$$\sum_{i=1}^{s} g(i)^k = 1$$

to the solution of equations of the form

$$\sum_{i=1}^{s} g(i)^{k+r} = 1$$

for a fixed $r \in \mathbb{N}$ under some special requirements of the generator $g : \mathbb{N} \longrightarrow \mathbb{R}$.

3. Application to solutions of the Erdős-Moser equation

In this section we apply the notion of the **olloid** to study solutions of the Erdős-Moser equation. We launch the following method as an outgrowth of Lemma 2.4.

Theorem 3.1 (The extension method)**.** *If the equation*

$$\sum_{n=1}^{s} n^k = (s+1)^k$$

has a solution and there exist some $r \in \mathbb{N}$ *such that*

$$1 - \frac{1}{g(s)^r} > \int_1^s \frac{g'(t)}{g(t)^2} dt + \frac{1}{g(s)} \int_1^s \frac{g'(t)}{g(t)^2} dt + \cdots + \frac{1}{g(s)^{r-1}} \int_1^s \frac{g'(t)}{g(t)^2} dt$$

with

$$g(i) := \frac{s+1-i}{(s+1)}$$

for $1 \leq i \leq s$*. Then the equation*

$$\sum_{n=1}^{s} n^{k+r} = (s+1)^{k+r}$$

also has a solution.

Proof. Suppose the equation

$$\sum_{n=1}^{s} n^k = (s+1)^k \tag{3.1}$$

has a solution. Then the equation (3.1) can be recast as

$$\sum_{n=1}^{s} \left(\frac{n}{s+1} \right)^k = 1 \tag{3.2}$$

which can also be transformed into the sum

$$\sum_{n=1}^{s} g(i)^k = 1$$

with

$$g(i) := \frac{s+1-i}{(s+1)}.$$

The function

$$g(i) := \frac{s+1-i}{(s+1)}$$

for $1 \leq i \leq s$ is decreasing and has continuous derivative on $[1, s]$ so that if there exists some $r \in \mathbb{N}$ such that

$$1 - \frac{1}{g(s)^r} > \int_1^s \frac{g'(t)}{g(t)^2}dt + \frac{1}{g(s)}\int_1^s \frac{g'(t)}{g(t)^2}dt + \cdots + \frac{1}{g(s)^{r-1}}\int_1^s \frac{g'(t)}{g(t)^2}dt$$

with

$$g(i) := \frac{s+1-i}{(s+1)}$$

for $1 \leq i \leq s$, then by appealing to Lemma 2.4 the equation

$$\sum_{n=1}^{s} g(i)^{k+r} = 1 \tag{3.3}$$

also has a solution. We note that the equation (3.4) can also be transformed to the equation

$$\sum_{n=1}^{s} \left(\frac{n}{s+1}\right)^{k+r} = 1 \tag{3.4}$$

so that it has a solution. Since equation (3.4) can be recast as

$$\sum_{n=1}^{s} n^{k+r} = (s+1)^{k+r}$$

the claim follows immediately. □

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References

1. Moser, Leo, *On the diophantine equation* $1^n + 2^n + 3^n + \cdots + (m-1)^n = m^n$., Scripta Math, vol. 19, 1953, 84–88.
2. Butske, William and Jaje, Lynda and Mayernik, Daniel, $\sum_{p|N} \frac{1}{p} + \frac{1}{N} = 1$*, pseudoperfect numbers, and perfectly weighted graphs.*, Mathematics of computation, vol. 69:229, 2000, 407–420.
3. Krzysztofek, B, *The equation* $1^n + ... + m^n = (m+1)^n$, Wyz. Szkol. Ped. w. Katowicech-Zeszyty Nauk. Sekc. Math, vol. 5, 1966, 47–54.
4. Moree, P and te Riele, HJJ and Urbanowicz, J, *Divisibility properties of integers* m*,* k *satisfying* $1^k + 2^k + \cdots + (m-1)^k = m^k$, Mathematics of Computation, vol. 63:208, 1994, 799–815.
5. Gallot, Yves and Moree, Pieter and Zudilin, Wadim, *THE ERDŐS-MOSER EQUATION* $1^k + 2^k + \cdots + (m-1)^k = m^k$ *REVISITED USING CONTINUED FRACTIONS*, Mathematics of Computation, vol. 63:208, JSTOR, 2011, 1221–1237.

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