On the Collatz Conjecture Problem

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Abstract

It is proved that

$$\lim_{n\to\infty} a_n = 1,$$

where a_n is the n^{th} term in the Collatz sequence:

$$a_{n+1} = (3a_n + 1)/\ 2^{\gamma_{n+1}}$$

where a_n is an odd positive integer and $2^{\gamma_{n+1}}$ is the highest power of two dividing $3a_n + 1$.

1. Introduction

Define a recursive function introduced by R. E Crandall[1] :

$$a_{n+1} = (3a_n + 1)/\ 2^{\gamma_{n+1}} \qquad (1)$$

where a_n is an odd positive integer and $2^{\gamma_{n+1}}$ is the highest power of two dividing $3a_n + 1$. The Collatz conjecture [1] asserts that for every odd positive integer a_0 and by applying eq.(1) , there exists k $\in$ N such that $a_k = 1$.

For example,

Let $a_0 = 3$ then $a_1 = 5$, $a_2 = 1$; $\gamma_1 = 1$, $\gamma_2 = 4$.

2. Validity of the Collatz conjecture

Start with an odd positive integer a_0 , a_n can be formulated as

$$a_n = \frac{(3a_0 + 1)3^{n-1}}{2^{(\gamma 1 + \gamma 2 + ------ + \gamma n)}} + \sum_{j=0}^{n-2} \frac{3^j}{2^{(\gamma_{n-j} + \gamma_{n-j+1} + ------ + \gamma_n)}} \qquad (2)$$

For an odd positive integer a_0 , the Collatz conjecture is true if we can find

k $\in$ N, and $\gamma_i \geq 1$, i = 1, 2,----, k such that $a_k = 1$.

Consider each term in Eq.(2) for n→ ∞, there are infinite i such that $\gamma_i = 2$, i.e. $\gamma_1 + \gamma_2 + --- + \gamma_m$ = c, $\gamma_{m+1} + \gamma_{m+2} + ---- + \gamma_n$ = 2(n-m) for large n $\gg m$.

$$\lim_{n\to\infty} \frac{(3a_0 + 1)3^{n-1}}{2^{(\gamma 1+\gamma 2+------+\gamma n)}} \leq \lim_{n\to\infty} \frac{(3a_0 + 1)3^{n-1}}{2\hat{}(m+2(n-m))} = 0$$

$$\lim_{n\to\infty} \sum_{j=0}^{n-2} \frac{3^j}{2^{(\gamma_{n-j}+\gamma_{n-j+1}+------+\gamma_n)}} = \lim_{n\to\infty} \frac{1}{2^{\gamma n}} + \frac{3}{2^{\gamma(n-1)+\gamma n}} + \frac{3^2}{2^{\gamma(n-2)} +\gamma(n-1)+\gamma n} + --------- + \frac{3^{n-2}}{2^{\gamma_2} + \gamma_3 + ---+ \gamma_n} \leq \lim_{n\to\infty} \frac{1}{4} \sum_{j=0}^{n-2} (\frac{3}{4})^j = 1.$$

Thus,

$$\lim_{n\to\infty} a_n \leq 1,$$

but a_n must be an odd integer, then $\lim_{n\to\infty} a_n = 1$.

3. Conclusion

In order to solve Eq. (2) for n and γ_i , i=1,2,---,n to have a_n =1, we must extend n to infinity and enforce condition for γ_i =2 for large i ≫ 1 in order to have $\lim_{n\to\infty} a_n = 1$.

References

[1] R . E. Crandall, " On the "3x+1" Problem, Mathematics of computation, volume 32, number 144, October 1978, page 1281-92.