A Simple Proof that Goldbach's Conjecture is True

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Abstract

A induction proof shows Goldbach's conjecture is correct. It is as simple as can be imagined. A table consisting of two rows is used. The lower row counts from 0 to any n and and the top row counts down from $2n$ to n. All columns will have all numbers that add to $2n$. Using a sieve, all composites are crossed out and only columns with primes are left. For $k = 5$, that's the $2 \cdot 5 = 10$ case, suppose that primes on the lower row always map to composites on the top and that this results in too many composites on the top. This is true for this base case. Suppose it is true for $k = n$, then the shifts and additions necessary for the $k = n + 1$ case maintain this property of too many composites on top. The contrapositive is that there exists a prime on the bottom that maps to a prime on top and Goldbach is established: the sum of these two primes is $2(n + 1)$.

Introduction

Hardy and Apostol spend some time on Goldbach's conjecture [1, 2]. The conjecture has it that every even number can be expressed as the sum of two primes. And indeed it is fascinating to try it on some even numbers and quickly find some instances.

Various angles for finding examples are possible. One can just add any two odd primes and the result will be even. So $3 + 5 = 8$, $5 + 7 = 10$, and so on. This will give lots of even sums fast. If one allows, which the conjecture does, non distinct primes then we can add $3+3 = 6$ and $5+5 = 10$ and start to sense that, indeed, you might just get all evens.

Thence to the central rub with this conjecture. You get lots and lots of pairs that sum to ever larger evens. A plethora of evidence starts accumulating and one can quickly lose sight of the goal of proving it is generally true. Things inevitably get complicated and the schemes get more and more elaborate; and annoyingly every now and again extremely simple. At least that has been my experience.

This article gives a simple scheme that leads to proof by induction that Goldbach's conjecture is true.

A Sieve

Given an even $2n$, we know $2 \ldots n$ has lots of early primes and $n+1 \ldots 2n$ has at least one prime per Bertrand's postulate [2]. Use two rows to count up to $2n$ with the lower row consisting of $0 \ldots n$ and the top row consisting of $n \ldots 2n$, counting up from right to left. Scratch off all but the first prime multiples of the first and second rows. What's left are primes on the first row and any primes on the second row seem to always line up with lower row primes. These pairs are odd primes that sum to $2n$.

Here's is an example: Table 1. This procedure is called a sieve. The Greek mathematician Eratosthenes used it to find primes. We are doing the same thing.

$\cancel{20}$	19	$\cancel{18}$	17	$\cancel{16}$	$\cancel{15}$	$\cancel{14}$	13	$\cancel{12}$	$\cancel{11}$	10
$\cancel{0}$	$\cancel{1}$	2	3	$\cancel{4}$	5	$\cancel{6}$	7	$\cancel{8}$	$\cancel{9}$	$\cancel{10}$

Table 1: A simple sieve process results in prime pairs adding to 20.

What would a counter-example look like? Instead of pairing primes with primes and thus preserving prime columns, a counter-example must consist of all primes on the first $0 \ldots n$ row pairing always with composites on the second $n \ldots 2n$ row.

Table 2 shows the result of applying this rule to the $2 \cdot 5 = 10$ base case. Some cells are coded: DC means doesn't count, C means composite; and CC means correctly composite. Comparing Table 1 with Table 2, this base case is established: this rule results in too many composites on the top row.

DC	DC	C	C	16	CC	14	C	12	11	10
DC	DC	2	3	4	5	6	7	8	9	10

Table 2: The base $2 \cdot 5 = 10$ base case for an induction argument.

Without loss of generality, we'll show that assuming this base 10 case has this property, that's our $n = k$ case (any case can be used), the $n = k+1$ case follows. Table 3 gives then the $2(5+1) = 12$ case. Assuming the $k = 5$ case, the number of composites between 10 and 20 is too many. We've indicated that this case applies with 5 subscripts. The F subscript indicates the fact that the top row number in this position has the same composite or prime status as the bottom row number at this position: they are the same number. Hence the status of the $k+1$ proposition is not changed by this cell. The other cells that could change the property of too many composites are the ones that don't count. Hence for this case applying the rule, there are too many composites on the top row. The shift doesn't change the status of the upper row: composites by hypotheses are still composites.

22_{DC}	21_{DC}	20_5	19_5	18_5	17_5	16_5	15_5	14_5	13_5	12_5	11_F
0_{DC}	1_{DC}	2	3	4	5	6	7	8	9	10	11_F

Table 3: The $k+1$ case with the k case applied.

The contrapositive of this rule is there exists a prime in the lower row that maps to a prime on the upper, but that is Goldbach's conjecture: it's true.

References

[1] Apostol, T. M. (1976). *Introduction to Analytic Number Theory*. New York: Springer.

[2] Hardy, G. H., Wright, E. M., Heath-Brown, R. , Silverman, J. , Wiles, A. (2008). *An Introduction to the Theory of Numbers*, 6th ed. London: Oxford Univ. Press.