Ramanujan, Integral, Continued Fraction

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Abstract. Ramanujan (1887-1920), notable achievements in the evaluation of integrals.

1. **Introduction**.(Ramanujan's letters to G.H. Hardy):

$$4\int_0^\infty \frac{x\,e^{-x\sqrt{5}}}{\cosh x}dx=\frac{1}{1+}\frac{1^2}{1+}\frac{1^2}{1+}\frac{2^2}{1+}\frac{2^2}{1+}\frac{3^2}{1+}\frac{3^2}{1+\cdots}$$

In this note we give some formulas related to $\int_0^\infty \frac{x\,e^{-x\sqrt{5}}}{\cosh x}dx$.

Recall that

$$\frac{\pi}{4}=\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}$$

$$\varphi=\frac{1+\sqrt{5}}{2}$$

φ is the Golden ratio.

2. Related Formulas

Entry 1.

$$\int_0^\infty \frac{x\,e^{-x\sqrt{5}}}{\cosh x}dx=\int_0^\infty x\,e^{-x(\sqrt{5}-1)}\,(1-\tanh x)\,dx$$

$$\int_0^\infty \frac{x\,e^{-x\sqrt{5}}}{\cosh x}dx=\int_0^\infty x\,e^{-x(\sqrt{5}+1)}\,(1+\tanh x)\,dx$$

Entry 2.

$$\int_0^\infty \frac{x\,e^{-x\sqrt{5}}}{\cosh x}dx=\frac{1}{\left(\sqrt{5}-1\right)^2}-\int_0^\infty x\,e^{-x(\sqrt{5}-1)}\tanh x\,dx$$

$$\int_0^\infty \frac{x\,e^{-x\sqrt{5}}}{\cosh x}dx=\frac{1}{\left(\sqrt{5}+1\right)^2}+\int_0^\infty x\,e^{-x(\sqrt{5}+1)}\tanh x\,dx$$

Entry 3.

$$\int_0^\infty \frac{x\,e^{-x\sqrt{5}}}{\cosh x}dx=\frac{1}{2}\int_0^1\frac{(1-x)^{1/\varphi}}{(1+x)^{\varphi}}\ln\left(\frac{1+x}{1-x}\right)dx$$

Entry 4.

$$\int_0^\infty \frac{x\,e^{-x\sqrt{5}}}{\cosh x}\,dx = \int_0^\infty \frac{\ln\left(x+\sqrt{1+x^2}\right)}{1+x^2}\left(x+\sqrt{1+x^2}\right)^{-\sqrt{5}}\,dx$$

$$\int_0^\infty \frac{x\,e^{-x\sqrt{5}}}{\cosh x}\,dx = \int_0^{\pi/2} (\sec x + \tan x)^{-\sqrt{5}} \ln(\sec x + \tan x)\,dx$$

Entry 5.

$$\int_0^\infty \frac{x\,e^{-x\sqrt{5}}}{\cosh x}\,dx = \frac{\pi}{2\sqrt{5}} - 2\int_0^\infty e^{-x\sqrt{5}} \tan^{-1}(e^{-x})\,dx - \frac{1}{\sqrt{5}}\int_0^\infty \frac{x\,e^{-x\sqrt{5}}\sinh x}{(\cosh x)^2}\,dx$$

$$\int_0^\infty \frac{x\,e^{-x\sqrt{5}}}{\cosh x}\,dx = \frac{\pi}{2\sqrt{5}} - 2\sum_{n=0}^\infty \frac{(-1)^n}{(2n+1)\left(2n+1+\sqrt{5}\right)} - \frac{1}{\sqrt{5}}\int_0^\infty \frac{x\,e^{-x\sqrt{5}}\sinh x}{(\cosh x)^2}\,dx$$

Entry 6.

$$\int_0^\infty \frac{x\,e^{-x\sqrt{5}}}{\cosh x}\,dx = -2\int_0^1 \frac{x^{\sqrt{5}}\ \ln x}{1+x^2}\,dx$$

$$\int_0^\infty \frac{x\,e^{-x\sqrt{5}}}{\cosh x}\,dx = 2\int_1^\infty \frac{x^{-\sqrt{5}}\ \ln x}{1+x^2}\,dx$$

Entry 7.

$$\int_0^\infty \frac{x\,e^{-x\sqrt{5}}}{\cosh x}\,dx$$

$$= \pi\,\frac{\ln 3}{3}\,\sqrt{3}^{-\sqrt{5}} - 2\int_1^{\sqrt{3}} \frac{1 - x\sqrt{5}}{x^{1+\sqrt{5}}}\tan^{-1} x\,dx$$

$$+ 2\sum_{n=0}^\infty \frac{(-1)^n \sqrt{3}^{-\left(2n+1+\sqrt{5}\right)}}{2n+1+\sqrt{5}}\left(\ln 3 + \frac{1}{2n+1+\sqrt{5}}\right)$$

Entry 8.

$$\int_0^\infty \frac{x\,e^{-x\sqrt{5}}}{\cosh x}\,dx = -2\int_0^{\pi/4} (\tan x)^{\sqrt{5}} \ln(\tan x)\,dx$$

$$\int_0^\infty \frac{x\,e^{-x\sqrt{5}}}{\cosh x}\,dx = -2\int_0^{\ln\left(1+\sqrt{2}\right)} (\sinh x)^{\sqrt{5}}\,\mathrm{sech}\,x\,\ln(\sinh x)\,dx$$

Entry 9.

$$\int_0^\infty \frac{x\,e^{-x\sqrt{5}}}{\cosh x}\,dx = \sum_{n=0}^\infty 2^{-n-2}\sum_{k=0}^n \binom{n}{k}\frac{(-1)^k}{(k+\varphi)^2}$$

$$\int_0^\infty \frac{x\,e^{-x\sqrt{5}}}{\cosh x}\,dx = \sum_{n=0}^\infty 3^{-n-1}\sum_{k=0}^n \binom{n}{k}\frac{(-2)^k}{(k+\varphi)^2}$$

Entry 10.

$$\int_0^\infty \frac{x\, e^{-x\sqrt{5}}}{\cosh x}\,dx = -\frac{1}{2}\int_0^1 \frac{x^{1/\varphi}\ln x}{1+x}\,dx$$

$$\int_0^\infty \frac{x\, e^{-x\sqrt{5}}}{\cosh x}\,dx = -\frac{\varphi^2}{2}\int_0^1 \frac{x^{\varphi}\ln x}{1+x^{\varphi}}\,dx$$

Entry 11.

$$\int_0^\infty \frac{x\, e^{-x\sqrt{5}}}{\cosh x}\,dx = \frac{\pi^2}{24} + \frac{\varphi^2}{2}\int_0^1 \frac{x^{1/\varphi} - x^{\varphi}}{1+x^{\varphi}}\ln x\,dx$$

$$\int_0^\infty \frac{x\, e^{-x\sqrt{5}}}{\cosh x}\,dx = \frac{\pi^2}{24} - \frac{\varphi^2}{2}\int_0^\infty \frac{e^{-x/\varphi} - e^{-x\,\varphi}}{1+e^{-x\,\varphi}}\,xe^{-x}\,dx$$

Entry 12.

$$\int_0^\infty \frac{x\, e^{-x\sqrt{5}}}{\cosh x}\,dx = \frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n\,(n-1+\varphi)} + \frac{1}{2\varphi}\int_0^1 \frac{\ln(1+x)\ln x}{x^{2-\varphi}}\,dx$$

Entry 13. For $0 < a < 1$ we have

$$\int_0^\infty \frac{x\, e^{-x\sqrt{5}}}{\cosh x}\,dx$$

$$= \frac{1}{2}\ln\frac{1}{a}\sum_{n=0}^{\infty}\frac{(-1)^n a^{n+\varphi}}{n+\varphi} + \frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n a^{n+\varphi}}{(n+\varphi)^2}$$

$$+\sum_{n=0}^{\infty}\frac{1}{n+2}\left(\frac{1-a}{2}\right)^{n+2}\sum_{m=0}^{n}\frac{2^m}{m+1}\sum_{k=0}^{n-m}\frac{2^k(-1/\varphi)_k}{k!}$$

$(x)_k$ is the Pochhammer symbol.

For $a = 1/2$ we have

$$\int_0^\infty \frac{x\, e^{-x\sqrt{5}}}{\cosh x}\,dx$$

$$= \frac{\ln 2}{2}\sum_{n=0}^{\infty}\frac{(-1)^n\, 2^{-n-\varphi}}{n+\varphi} + \frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n\, 2^{-n-\varphi}}{(n+\varphi)^2}$$

$$+\sum_{n=0}^{\infty}\frac{2^{-2n-4}}{n+2}\sum_{m=0}^{n}\frac{2^m}{m+1}\sum_{k=0}^{n-m}\frac{2^k(-1/\varphi)_k}{k!}$$

Entry 14.

$$\int_0^\infty \frac{x\, e^{-x\sqrt{5}}}{\cosh x}\,dx = \frac{\pi^2}{24} - \frac{1}{2\varphi}\sum_{n=0}^{\infty}\frac{(-1)^n(2n+1+\varphi)}{(n+1)^2(n+\varphi)^2}$$

$$\frac{\pi^2}{6} = \frac{2}{\varphi}\sum_{n=0}^{\infty}\frac{(-1)^n(2n+1+\varphi)}{(n+1)^2(n+\varphi)^2} + \frac{1}{1+}\frac{1^2}{1+}\frac{1^2}{1+}\frac{2^2}{1+}\frac{2^2}{1+}\frac{3^2}{1+}\frac{3^2}{1+\cdots}$$

Entry 15.

$$\int_0^\infty \frac{x\,e^{-x\sqrt{5}}}{\cosh x}\,dx = \frac{1}{5} - \frac{1}{2}\sum_{n=0}^{\infty} \frac{(-1)^n(12(n+\varphi)^2-1)}{(n+\varphi)^2(4(n+\varphi)^2-1)^2}$$

$$\frac{4}{5} - 2\sum_{n=0}^{\infty} \frac{(-1)^n(12(n+\varphi)^2-1)}{(n+\varphi)^2(4(n+\varphi)^2-1)^2} = \frac{1}{1+}\frac{1^2}{1+}\frac{1^2}{1+}\frac{2^2}{1+}\frac{2^2}{1+}\frac{3^2}{1+}\frac{3^2}{1+\cdots}$$

Entry 16.

$$\int_0^\infty \frac{x\,e^{-x\sqrt{5}}}{\cosh x}\,dx = 2G - \frac{\sqrt{5}}{2}\sum_{n=0}^{\infty} \frac{(-1)^n(4n+1+2\varphi)}{(2n+1)^2(n+\varphi)^2}$$

$$8G = 2\sqrt{5}\sum_{n=0}^{\infty} \frac{(-1)^n(4n+1+2\varphi)}{(2n+1)^2(n+\varphi)^2} + \frac{1}{1+}\frac{1^2}{1+}\frac{1^2}{1+}\frac{2^2}{1+}\frac{2^2}{1+}\frac{3^2}{1+}\frac{3^2}{1+\cdots}$$

$G = \sum_{n=0}^{\infty}(-1)^n(2n+1)^{-2}$ is the Catalan constant.

Entry 17.

$$\int_0^\infty \frac{x\,e^{-x\sqrt{5}}}{\cosh x}\,dx = 2G - 8\int_0^\infty x e^{-x\sqrt{5}}\sinh\left(x\sqrt{5}\right)\operatorname{sech}(2x)\,dx$$

$$= 2G - 8\int_0^\infty \frac{x\operatorname{sech}(2x)}{1+\coth\left(x\sqrt{5}\right)}\,dx = 2G - 8\int_0^\infty \frac{x\tanh\left(x\sqrt{5}\right)\operatorname{sech}(2x)}{1+\tanh\left(x\sqrt{5}\right)}\,dx$$

where $G = \sum_{n=0}^{\infty}(-1)^n(2n+1)^{-2}$ is the Catalan constant.

Entry 18.

$$\int_0^\infty \frac{x\,e^{-x\sqrt{5}}}{\cosh x}\,dx$$

$$= \frac{1-(1+2\varphi)e^{-2\varphi}}{4\varphi^2} + \int_0^1 x\,e^{-2\varphi x}\tanh x\,dx$$

$$+ \frac{e^{-2\varphi}}{2}\sum_{n=0}^{\infty} \frac{(-1)^n(2n+2\varphi+1)e^{-2n}}{(n+\varphi)^2}$$

Entry 19.

$$\int_0^\infty \frac{x\,e^{-x\sqrt{5}}}{\cosh x}\,dx$$

$$= 2e^{-2\varphi}\sum_{n=0}^{\infty} \frac{(-1)^n(2n+2\varphi+1)e^{-2n}}{(2n+2\varphi)^2}$$

$$+ \sum_{n=0}^{\infty} 2^{-n}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\left(\frac{1-(2k+2\varphi+1)e^{-2k-2\varphi}}{(2k+2\varphi)^2}\right)$$

Entry 20.

$$\int_0^\infty \frac{x\,e^{-x\sqrt{5}}}{\cosh x}\,dx = \frac{1}{2}\sum_{n=0}^{\infty} \frac{(-1)^n e^{-n-\varphi}}{(n+\varphi)\sinh(n+\varphi)} + \sum_{n=0}^{\infty} e^{-n\sqrt{5}}\int_0^1 \frac{x\,e^{-x\sqrt{5}}}{\cosh(x+n)}\,dx$$

Entry 21.

$$\int_0^\infty \frac{x\, e^{-x\sqrt{5}}}{\cosh x}\, dx = \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n\,(n+1)\varphi^{-n}(1-2^{-n-1})\,\zeta(n+2)$$

$$\int_0^\infty \frac{x\, e^{-x\sqrt{5}}}{\cosh x}\, dx = \frac{1}{2\varphi^2}-\frac{1}{2\varphi^4}+\frac{1}{2}\sum_{n=0}^{\infty}(-1)^n\,(n+1)\varphi^{n}\big(1-(1-2^{-n-1})\zeta(n+2)\big)$$

where $\zeta(x)=\sum_{k=1}^{\infty}k^{-x}$, $x>0$, is the Riemann zeta function.

Entry 22.

$$\int_0^\infty \frac{x\, e^{-x\sqrt{5}}}{\cosh x}\, dx = \int_{-\infty}^{\infty} e^{2x-\sqrt{5}\,e^{x}}\ \operatorname{sech}(e^{x})\, dx$$

$$\int_0^\infty \frac{x\, e^{-x\sqrt{5}}}{\cosh x}\, dx = \int_{-\infty}^{\infty} e^{-2x-\sqrt{5}\,e^{-x}}\ \operatorname{sech}(e^{-x})\, dx$$

Entry 23.

$$\int_0^\infty \frac{x\, e^{-x\sqrt{5}}}{\cosh x}\, dx$$

$$= \frac{1}{\varphi}\sum_{n=1}^{\infty} n\, e^{-n(\sqrt{5}+1)}\,\big(F(1,\varphi,1+\varphi,-e^{-2n})$$

$$- e^{-2\varphi}F(1,\varphi,1+\varphi,-e^{-2n-2})\big)$$

$$+ 2\sum_{n=0}^{\infty}\int_0^1 \frac{x}{e^{(x+n)(\sqrt{5}+1)}+e^{(x+n)(\sqrt{5}-1)}}\, dx$$

where $F(a,b,c,x)$ is the Gauss hyergeometric function.

Entry 24.

$$\int_0^\infty \frac{x\, e^{-x\sqrt{5}}}{\cosh x}\, dx = \varphi^2\int_0^\infty x\, e^{-2x}\,(1-\tanh(\varphi\, x))\, dx$$

$$\int_0^\infty \frac{x\, e^{-x\sqrt{5}}}{\cosh x}\, dx = \frac{\varphi^2}{4}-\varphi^2\int_0^\infty x\, e^{-2x}\ \tanh(\varphi\, x)\, dx$$

Entry 25.

$$\int_0^\infty \frac{x\, e^{-x\sqrt{5}}}{\cosh x}\, dx = \varphi^2\int_0^\infty \frac{x\, e^{-(2+\varphi)x}}{\cosh(\varphi\, x)}\, dx$$

Entry 26.

$$\int_0^\infty \frac{x\, e^{-x\sqrt{5}}}{\cosh x}\, dx = 2\varphi^2\int_0^\infty \frac{x\, e^{-2x}}{1+e^{2\varphi\, x}}\, dx$$

Entry 27.

$$\int_0^\infty \frac{x\, e^{-x\sqrt{5}}}{\cosh x}\, dx = \frac{1}{4}\int_0^\infty \frac{x\,\left(e^{-x/\varphi}-e^{-x\,\varphi}\right)}{\sinh x}\, dx$$

$$\int_0^\infty \frac{x\, e^{-x\sqrt{5}}}{\cosh x}\, dx = \frac{1}{4}\int_0^\infty \frac{x\,\left(e^{-x/\varphi}+e^{-x\,\varphi}\right)}{1+\cosh x}\, dx$$

Entry 28.

$$\int_0^\infty \frac{x\, e^{-x\sqrt{5}}}{\cosh x}\,dx = \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{1}{(2n+\varphi)^2} - \frac{1}{(2n+1+\varphi)^2}\right) = \frac{1}{2}\sum_{n=0}^{\infty}\frac{4n+2\varphi+1}{(2n+\varphi)^2(2n+1+\varphi)^2}$$

Entry 29.

$$\int_0^\infty \frac{x\, e^{-x\sqrt{5}}}{\cosh x}\,dx = \int_0^\infty \frac{x}{\sinh x}\left(e^{-x\sqrt{5}} - \frac{1}{2}e^{-x\,\varphi}\right)dx$$

$$\int_0^\infty \frac{x\, e^{-x\sqrt{5}}}{\cosh x}\,dx = \int_0^\infty \frac{x}{\sinh x}\left(\frac{1}{2}e^{-x/\varphi} - e^{-x\sqrt{5}}\right)dx$$

Entry 30.

$$\int_0^\infty \frac{x\, e^{-x\sqrt{5}}}{\cosh x}\,dx$$

$$= \frac{1}{4\varphi^2}$$

$$+ \sum_{n=1}^{\infty} 2^{-n-2}\left(\sum_{k=0}^{[n/2]}\binom{n}{2k}(2k+\varphi)^{-2} - \sum_{k=0}^{[(n-1)/2]}\binom{n}{2k+1}(2k+1+\varphi)^{-2}\right)$$

Entry 31.

$$\int_0^\infty \frac{x\, e^{-x\sqrt{5}}}{\cosh x}\,dx = \frac{1}{2}\sum_{n=1}^{\infty}\frac{(n-1)!}{n\,(\varphi)_n} - \sum_{n=0}^{\infty}\frac{1}{(2n+1+\varphi)^2}$$

$$\int_0^\infty \frac{x\, e^{-x\sqrt{5}}}{\cosh x}\,dx = \sum_{n=0}^{\infty}\frac{1}{(2n+\varphi)^2} - \frac{1}{2}\sum_{n=1}^{\infty}\frac{(n-1)!}{n\,(\varphi)_n}$$

$(x)_n$ is the Pochhammer symbol.

3. Endnote

Entry 32.

$$2\sum_{n=1}^{\infty}\frac{n\, e^{-n\sqrt{5}}}{\cosh n} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(\sinh(n+\varphi))^2}$$

Entry 33. For $N = 1,2,3,\ldots$ we have

$$\int_0^N \frac{x\, e^{-x\sqrt{5}}}{\cosh x}\,dx$$

$$= \sum_{n=1}^{N}\frac{n\, e^{-n\sqrt{5}}}{\cosh n} - \int_0^N \frac{e^{-x\sqrt{5}}\left(1 - x\sqrt{5} - x\ \tanh x\right)}{\cosh x}\left(x - [x] - \frac{1}{2}\right)dx$$

$$- \frac{N\, e^{-N\sqrt{5}}}{2\cosh N}$$

4. Future Research

Entry 34. For $n = 1,2,3,\ldots$ we have

$$\frac{F_{2n}^2}{2}\int_0^\infty \frac{x\,e^{-xF_{2n+1}}}{1+e^{-xF_{2n}}}dx < \int_0^\infty \frac{x\,e^{-x\sqrt{5}}}{\cosh x}dx < \frac{F_{2n-1}^2}{2}\int_0^\infty \frac{x\,e^{-xF_{2n}}}{1+e^{-xF_{2n-1}}}dx$$

where F_n is the Fibonacci number:

$$\{F_n : n \geq 1\} = \{1,1,2,3,5,8,13,21,\ldots\}$$

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