

# Fine-structure constant from the madelung constant

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## Abstract

A new interpretation and a very accurate value of the fine-structure constant has been discovered in terms of the madelung constant  $b_2(2)$ . The equation is simple, elegant and symmetrical in a great physical meaning. This accurate expression is the most impressive since it is simple and contains just a few prime numbers and the madelung constant. These prime numbers can be possibly connected to finite groups. Also we will present the continued fraction for the fine-structure constant.

## Keywords

Fine-structure constant , Dimensionless physical constants , Madelung constant , Archimedes constant

## 1. Introduction

In physics, a dimensionless physical constant is a physical constant that is dimensionless, a pure number having no units attached and having a numerical value that is independent of whatever system of units may be used. The term fundamental physical constant is used to refer to some universal dimensionless constants. A long-sought goal of theoretical physics is to find first principles from which all of the fundamental dimensionless constants can be calculated and compared to the measured values. In the 1920s and 1930s, Arthur Eddington embarked upon extensive mathematical investigation into the relations between the fundamental quantities in basic physical theories, later used as part of his effort to construct an overarching theory unifying quantum mechanics and cosmological physics. The mathematician Simon Plouffe has made an extensive search of computer databases of mathematical formulas, seeking formulas for the mass ratios of the fundamental particles. An empirical relation between the masses of the electron, muon and tau has been discovered by physicist Yoshio Koide, but this formula remains unexplained.

Dimensionless physical constants cannot be derived and have to be measured. Developments in physics may lead to either a reduction or an extension of their number: discovery of new particles, or new relationships between physical phenomena, would introduce new constants, while the development of a more fundamental theory might allow the derivation of several constants from a more fundamental constant. A long-sought goal of theoretical physics is to find first principles from which all of the fundamental dimensionless constants can be calculated and compared to the measured values. The laws of physics have a set of fundamental constants, and it is generally admitted that only dimensionless combinations of constants have physical significance. These combinations include the electromagnetic and gravitational fine structure, along with the ratios of elementary particles masses. Cosmological measurements clearly depend on the values of these constants in the past and can therefore give information on their time dependence if the effects of time-varying constants can be separated from the effects of cosmological parameters.

The fine-structure constant is one of the most fundamental constants of physics. It describes the strength of the force of electromagnetism between elementary particles in what is known as the standard model of particle physics. In particular, the fine-structure constant sets the strength of electromagnetic interaction between light (photons) and charged elementary particles such as electrons and muons. The quantity  $\alpha$  was introduced into physics by A. Sommerfeld in 1916 and in the past has often been referred to as the Sommerfeld fine-structure constant. In order to explain the observed splitting or fine structure of the energy levels of the hydrogen atom, Sommerfeld extended the Bohr theory to include elliptical orbits and the relativistic dependence of mass on velocity.

One of the most important numbers in physics is the fine-structure constant  $\alpha$  which defines the strength of the electro-magnetic field. It is a dimensionless number independent of how we define our units of mass, length, time or electric charge. A change in these units of measurement leaves the dimensionless constant unchanged. The number can be seen as the chance that an electron emits or absorbs a photon. It's a pure number that shapes the universe to an

astonishing degree. Paul Dirac considered the origin of the number «the most fundamental unsolved problem of physics». The constant is everywhere because it characterizes the strength of the electromagnetic force affecting charged particles such as electrons and protons. Many eminent physicists and philosophers of science have pondered why  $\alpha$  itself has the value that it does, because the value shows up in so many important scenarios and aspects of physics. Nobody has come up with any ideas that are even remotely convincing. A similar situation occurs with the proton-electron mass ratio  $\mu$ , not because of its ubiquity, but rather how chemistry can be based on two key electrically charged particles of opposite electric charge that are opposite but of seemingly identical magnitude while their masses have a ratio that is quite large yet finite. These two questions have a huge bearing on why physics and chemistry behave the way they do. The product of the two quantities appears, at least at first glance, not to be so important. The fine-structure constant  $\alpha$  is defined as:

$$\alpha = \frac{q_e^2}{4\pi\epsilon_0\hbar c}$$

The 2018 CODATA recommended value of fine-structure constant is:

$$\alpha = 0,0072973525693(11)$$

With standard uncertainty  $0,0000000011 \times 10^{-3}$  and relative standard uncertainty  $1,5 \times 10^{-10}$ . For reasons of convenience, historically the value of the reciprocal of the fine-structure constant is often specified. The 2018 CODATA recommended value of  $\alpha^{-1}$  is given by:

$$\alpha^{-1} = 137,035999084(21)$$

With standard uncertainty  $0,000000021 \times 10^{-3}$  and relative standard uncertainty  $1,5 \times 10^{-10}$ . There is general agreement for the value of  $\alpha$ , as measured by these different methods. The preferred methods in 2019 are measurements of electron anomalous magnetic moments and of photon recoil in atom interferometry. The most precise value of  $\alpha$  obtained experimentally (as of 2012) is based on a measurement of  $g$  using a one-electron so-called "quantum cyclotron" apparatus, together with a calculation via the theory of QED that involved 12.672 tenth-order Feynman diagrams:

$$\alpha^{-1} = 137,035999174(35)$$

This measurement of  $\alpha$  has a relative standard uncertainty of  $2,5 \times 10^{-10}$ . This value and uncertainty are about the same as the latest experimental results. Further refinement of this work were published by the end of 2.020, giving the value:

$$\alpha^{-1} = 137,035999206(11)$$

with a relative accuracy of 81 parts per trillion. Also the fine-structure constant is universal scaling factor:

$$\alpha = \frac{2\pi r_e}{\lambda_e} = \frac{\lambda_e}{2\pi\alpha_0} = \frac{r_e}{l_{pl}} \frac{m_e}{m_{pl}} = \sqrt{\frac{r_e}{\alpha_0}}$$

There is a dream, which, albeit more often not confessed, occupies the most secret aspirations of theoreticians and is that of reducing the various constants of Physics to simple formula involving integers and transcendent numbers. The fine-structure constant plays an important role in modern physics. Yet it continues to be a mystery as to exactly what it represents and why it has the mystical value it has. The purpose of many sciences is to find the most accurate mathematical formula that can be found in the experimental value of fine-structure constant. Attempts to find a mathematical basis for this dimensionless constant have continued up to the present time. However, no numerological explanation has ever been accepted by the physics community.

## 2. The search for mathematical expression for the fine-structure constant

The mystery about the fine-structure constant is actually a double mystery. The first mystery – the origin of its

numerical value – has been recognized and discussed for decades. The second mystery – the range of its domain – is generally unrecognized.

— M. H. MacGregor (2.007). The Power of Alpha.

When I die my first question to the Devil will be: What is the meaning of the fine structure constant?

— Wolfgang Pauli

“God is a pure mathematician!” declared British astronomer Sir James Jeans. The physical Universe does seem to be organized around elegant mathematical relationships. And one number above all others has exercised an enduring fascination for physicists: 137,0359991.... It is known as the fine-structure constant and is denoted by the Greek letter alpha ( $\alpha$ ).”

— Paul Davies

“While twentieth-century physicists were not able to identify any convincing mathematical constants underlying the fine structure, partly because such thinking has normally not been encouraged, a revolutionary suggestion was recently made by the Czech physicist Raji Heyrovska, who deduced that the fine structure constant, ...really is defined by the [golden] ratio ....”

— Carl Johan Calleman, The Purposeful Universe: How Quantum Theory and Mayan Cosmology Explain the Origin and Evolution of Life

The fine-structure constant plays an important role in modern physics. Yet it continues to be a mystery as to exactly what it represents and why it has the mystical value it has. The elementary charge of electron  $e$  was proposed by Stoney in 1894 and discovered by Thomson in 1896, then Planck introduced the energy quanta  $h \cdot \nu$  in 1901 and explained it as photon  $E = h \cdot \nu$  by Einstein in 1905. Planck first noticed in 1905 that  $e^2/c$  and  $h$  have the same dimension. In 1909, Einstein found that there are two fundamental velocities in physics:  $c$  and  $e^2/h$  requiring explanation. He said, “It seems to me that we can conclude from  $h = e^2/c$  that the same modification of theory that contains the elementary quantum  $e$  as a consequence, will also contain as a consequence the quantum structure of radiation.” Albert Einstein was the first to use a mathematical formula for the fine-structure constant  $\alpha$  in 1909. This expression is:

$$\alpha = \frac{7\pi}{3.000}$$

with numerical value  $\alpha = 0,00733038286$  with an error accuracy of 0,45%. Later many scientists used other mathematical formulas for fine-structure constant but they are not at all accurate. These are Jeans 1913, Lewis Adams 1914, Lunn in 1922, Peirles in 1928 and others. Arthur Eddington was the first to focus on its inverse value and suggested that it should be an integer, that the theoretical value is  $\alpha^{-1} = 136$ . In his original document 1929 he applied the value:

$$\alpha^{-1} = 16 + 1/2 \times 16 \times (16 - 1) = 136$$

However, the experiments themselves consistently showed that  $\alpha^{-1} = 137$ . This forced him to look for an error in his original theory. He soon came to the conclusion that:

$$\alpha^{-1} = 137$$

He thus argued that the extra unit was a consequence of the initial exclusion of every elementary particle pair in the universe. In the document of 1929, Eddington considered that fine-structure constant relates in a simple way to the cosmological constants, as given by the expression:

$$\alpha = \frac{2\pi mc R_E}{h\sqrt{N}}$$

where  $N$  the cosmic number, the number of electrons and protons in the closed universe. Eddington always kept the

name and the symbol  $\alpha$ :

$$\alpha = \frac{hc}{2\pi q_e^2}$$

The first to find an exact formula for the fine-structure constant  $\alpha$  was the Swiss mathematician Armand Wyler in 1969. Based on the arguments concerning the congruent group, the group consists of simple Lorentz transformations such as the space-time dimensions that leave the Maxwell equations unchanged. The first form of the Wyler constant type is:

$$\alpha_w = \left( \frac{9}{16\pi^3} \right) \left( \frac{\pi}{5!} \right)^{\frac{1}{4}}$$

With numerical value  $\alpha_w=0,00729735252\dots$ . At the time it was proposed, they agreed with the experiment to be within 1.5 ppm for the value  $\alpha^{-1}$ .

We proposed in [8] the exact formula for the fine-structure constant  $\alpha$  in terms of the golden angle, the relativity factor and the fifth power of the golden mean:

$$\alpha^{-1} = 360 \cdot \varphi^{-2} - 2 \cdot \varphi^{-3} + (3 \cdot \varphi)^{-5} \quad (1)$$

with numerical value:

$$\alpha^{-1} = 137,035999164\dots$$

Another beautiful forms of the equations are:

$$\frac{1}{\alpha} = \frac{360}{\varphi^2} - \frac{2}{\varphi^3} + \frac{1}{3^5 \varphi^5} \quad (2)$$

$$\frac{1}{\alpha} = \frac{360}{\varphi^2} - \frac{2}{\varphi^3} + \frac{3^{-5}}{\varphi^5} \quad (3)$$

$$\alpha^{-1} = (362 - 3^{-4}) \cdot \varphi^{-2} - (1 - 3^{-5}) \cdot \varphi^{-1} \quad (4)$$

$$\alpha^{-1} = (362 - 3^{-4}) + (3^{-4} + 2 \cdot 3^{-5} - 364) \cdot \varphi^{-1} \quad (5)$$

$$\alpha^{-1} = 1 - 2 \cdot \varphi^{-1} + 360 \cdot \varphi^{-2} - \varphi^{-3} + (3 \cdot \varphi)^{-5} \quad (6)$$

#### 4. Fine-structure constant from the Madelung constant

Archimedes constant  $\pi$  is a mathematical constant that appears in many types in all fields of mathematics and physics. It is also found in various types from other disciplines, such as Cosmology, numbers, Statistics, fractals, thermodynamics, engineering, and electromagnetism. Archimedes constant  $\pi$  appears in the cosmological constant, Heisenberg's uncertainty principle, Einstein's field equation of general relativity, Coulomb's law for the electric force in vacuum, Magnetic permeability of free space, Period of a simple pendulum with small amplitude, Kepler's third law of planetary motion, the buckling formula, etc. In [7] we presented exact and approximate expressions between the Archimedes constant  $\pi$ , the golden ratio  $\varphi$ , the Euler's number  $e$  and the imaginary number  $i$ .

We proposed in [9] the simple and absolutely accurate expression for the fine-structure constant in terms of the Archimedes constant  $\pi$ :

$$\alpha^{-1} = \frac{2.706}{43} \pi \ln 2 \quad (7)$$

with absolutely accurate numerical value:

$$\alpha^{-1}=(0,00729735256959\dots)^{-1}=137,035999078\dots$$

The equivalent expression for the fine-structure constant is:

$$\alpha^{-1}=2\cdot 3\cdot 11\cdot 41\cdot 43^{-1}\cdot \pi\cdot \ln 2 \quad (8)$$

Other equivalent expression for the fine-structure constant are:

$$\alpha^{-1}=[(14^3-38)\cdot 43^{-1}]\cdot \ln 2\cdot \pi \quad (9)$$

$$\frac{2}{43}\times 3\times 11\times 41\pi\log(2)=\frac{2706}{43}\pi\log(a)\log_a(2) \quad (10)$$

$$\frac{2}{43}\times 3\times 11\times 41\pi\log(2)=\frac{5412}{43}\pi\coth^{-1}(3) \quad (11)$$

$$\frac{2}{43}\times 3\times 11\times 41\pi\log(2)=\frac{5412}{43}\pi\tanh^{-1}\left(\frac{1}{3}\right) \quad (12)$$

$$\frac{2}{43}\times 3\times 11\times 41\pi\log(2)=\frac{5412}{43}i\pi\cot^{-1}(3i) \quad (13)$$

$$\frac{2}{43}\times 3\times 11\times 41\pi\log(2)=-\frac{2706}{43}\pi S_{0,1}(-1) \quad (14)$$

$$\frac{2}{43}\times 3\times 11\times 41\pi\log(2)=\frac{5412}{43}i\pi\tan^{-1}\left(-\frac{i}{3}\right) \quad (15)$$

$$\frac{2}{43}\times 3\times 11\times 41\pi\log(2)=\frac{2706\pi}{43}\int_1^2\frac{1}{t}dt \quad (16)$$

$$\frac{2}{43}\times 3\times 11\times 41\pi\log(2)=-\frac{1353i}{43}\int_{-i\infty+\gamma}^{i\infty+\gamma}\frac{\Gamma(-s)^2\Gamma(1+s)}{\Gamma(1-s)}ds \text{ for } -1 < \gamma < 0 \quad (17)$$

This accurate expression is the most impressive since it is simple and contains just a few prime numbers and the Archimedes constant. These prime numbers can be possibly connected to finite groups (Group of Lie type).

The series representations for the fine-structure constant are:

$$\begin{aligned} \frac{2}{43}\times 3\times 11\times 41\log(2)\pi &= \frac{5412}{43}i\pi^2\left[\frac{\arg(2-x)}{2\pi}\right]+ \\ &\frac{2706}{43}\pi\log(x)-\frac{2706}{43}\pi\sum_{k=1}^{\infty}\frac{(-1)^k(2-x)^kx^{-k}}{k} \text{ for } x < 0 \end{aligned} \quad (18)$$

$$\begin{aligned} \frac{2}{43}\times 3\times 11\times 41\log(2)\pi &= \frac{2706}{43}\pi\left[\frac{\arg(2-z_0)}{2\pi}\right]\log\left(\frac{1}{z_0}\right)+\frac{2706}{43}\pi\log(z_0)+ \\ &\frac{2706}{43}\pi\left[\frac{\arg(2-z_0)}{2\pi}\right]\log(z_0)-\frac{2706}{43}\pi\sum_{k=1}^{\infty}\frac{(-1)^k(2-z_0)^kz_0^{-k}}{k} \end{aligned} \quad (19)$$

$$\frac{2}{43} \times 3 \times 11 \times 41 \log(2) \pi = \frac{5412}{43} i \pi^2 \left[ \frac{\pi - \arg\left(\frac{1}{z_0}\right) - \arg(z_0)}{2\pi} \right] + \frac{2706}{43} \pi \log(z_0) - \frac{2706}{43} \pi \sum_{k=1}^{\infty} \frac{(-1)^k (2 - z_0)^k z_0^{-k}}{k} \quad (20)$$

We pose:

$$ab = 2.706/43 = 2 \cdot 3 \cdot 11 \cdot 41 \cdot 43^{-1}$$

This combination of prime numbers we will call the coefficient for the fine-structure constant. So the expression for the fine-structure constant can be written as:

$$a^{-1} = -ab \cdot \pi \cdot \ln 2 \quad (21)$$

The repeating decimal of the coefficient for the fine-structure constant  $ab$  is:

$$\overline{62,930232558139534883720} \text{ (period 21)}$$

The continued fraction for the  $ab$  is:

$$62 + \frac{1}{1 + \frac{1}{13 + \frac{1}{3}}}$$

The Egyptian fraction expansion for the  $ab$  is:

$$62 + \frac{1}{2} + \frac{1}{3} + \frac{1}{11} + \frac{1}{167} + \frac{1}{473946}$$

The repeating decimal of the  $ab^{-1}$  is:

$$\overline{0,01589061345} \text{ (period 10)}$$

So from (21) the expression for the fine-structure constant with the Archimedes constant is:

$$a^{-1} = -ab \cdot \pi \cdot \ln 2$$

$$a \cdot \pi \cdot \ln 2 = -\overline{0,01589061345} \quad (22)$$

The Madelung constant is used in determining the electrostatic potential of a single ion in a crystal by approximating the ions by point charges. It is named after Erwin Madelung, a German physicist. Because the anions and cations in an ionic solid attract each other by virtue of their opposing charges, separating the ions requires a certain amount of energy. This energy must be given to the system in order to break the anion-cation bonds. The energy required to break these bonds for one mole of an ionic solid under standard conditions is the lattice energy. The Madelung constant is also a useful quantity in describing the lattice energy of organic salts. Izgorodina and coworkers have described a generalized method (called the EUGEN method) of calculating the Madelung constant for any crystal structure.

The quantities obtained [4] from cubic, hexagonal, etc. lattice sums, evaluated at  $s=1$ , are called Madelung constants. For cubic lattice sums:

$$b_n(2s) \equiv \sum_{k_1, \dots, k_n = -\infty}^{\infty} \frac{(-1)^{k_1 + \dots + k_n}}{(k_1^2 + \dots + k_n^2)^s},$$

The Madelung constants are expressible in closed form for even indices n, a few examples of which are summarized in the following table, where beta(n) is the Dirichlet beta function and eta(n) is the Dirichlet eta function. To obtain the closed form for b2(s), break up the double sum into pieces that do not include i=j=0:

$$b_2(2s) = 4 \left[ \sum_{i,j=1}^{\infty} \frac{(-1)^{i+j}}{(i^2 + j^2)^s} + \sum_{i=1}^{\infty} \frac{(-1)^i}{i^{2s}} \right].$$

The second of these sums can be done analytically as:

$$\sum_{i=1}^{\infty} \frac{(-1)^i}{i^{2s}} = -4^s (4^s - 2) \zeta(2s),$$

which in the case s=1 reduces to:

$$\sum_{i=1}^{\infty} \frac{(-1)^i}{i^2} = -\frac{1}{12} \pi^2.$$

The first sum is more difficult, but in the case s=1 can be written:

$$\sum_{i,j=1}^{\infty} \frac{(-1)^{i+j}}{i^2 + j^2} = \frac{1}{12} \pi (\pi - 3 \ln 2).$$

Combining these then gives the original sum as:

$$b_2(2) = -\pi \ln 2.$$

So from (7) the equivalent expressions for the fine-structure constant with the madelung constant b2(2) are:

$$\alpha^{-1} = -\frac{2.706}{43} b_2(2) \tag{23}$$

$$\alpha^{-1} = -2 \cdot 3 \cdot 11 \cdot 41 \cdot 43^{-1} \cdot b_2(2) \tag{24}$$

$$\alpha^{-1} = -[(14^3 - 38) \cdot 43^{-1}] \cdot b_2(2) \tag{25}$$

Also the expression for the fine-structure constant can be written as:

$$\alpha^{-1} = -ab \cdot b_2(2)$$

So from the expression for the fine-structure constant with the madelung constant b2(2) is:

$$a \cdot b_2(2) = -0,01589061345 \tag{26}$$

This accurate expression is the most impressive since it is simple and contains just a few prime numbers and the madelung constant. These prime numbers can be possibly connected to finite groups (Group of Lie type) This combination of prime numbers we will call the coefficient for the fine-structure constant.

## 5. Continued fraction for the fine-structure constant

The pattern of the continued fraction for the fine-structure constant is:



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