Bernoulli sums of powers and proof that Riemann Hypothesis is true

Eng. Marco Rolando Burgos Chambi

Jun 29th 2022

Abstract

On 1859, the german mathematician Georg Friedrich Bernhard Riemann made one of his most famous publications "On the Number of Prime Numbers less than a Given Quantity" when he was developing his explicit formula to give an exact number of primes less than a given number x, in which he conjectured that "all non-trivial zeros of the zeta function have a real part equal to $\frac{1}{2}$". Riemann was sure of his statement, but he could not prove it, remaining as one of the most important hypotheses unproven for 163 years.

In this paper, we have to prove that the Riemann Hypothesis is true, based on the Bernoulli power sum and its relation with the Riemann Zeta function.

Table of contents

1 Introduction **3**

1.1 Euler product and the sum of inverse powers. 3

1.2 The Euler-Riemann zeta function $\zeta(k)$. 3

2 Bernoulli numbers and the sum of k-th power. **4**

2.1 Obtaining the Bernoulli numbers by the sum of powers. 5

2.2 Simplified formula to find $S_k(n)$. 6

3 The funtion $S_k^*(n)$. **7**

3.1 Verification of the convergence of the function Δ_k. 8

4 Proof of the Riemann Hypothesis. **10**

5 Conclusion **13**

1 Introduction

In this article, we will resume the concepts necessary for the demonstration of the hypothesis of my previous publication, but without the need to consult them. It is also important to clarify that we will use the letter S to refer to the sum of Bernoulli powers, $S*$ a proposed function obtained from the expansion of the function S, and the letter k to refer to the variable of the zeta function instead of the usual letter s to avoid any confusion with the nomenclature, being k a complex number.

1.1 Euler product and the sum of inverse powers.

The infinite sum of inverse powers is a series of great interest for mathematics in number theory. Leonhard Euler [2] managed to relate this series to an infinite product that goes through all the prime numbers:

$$\sum_{n=1}^{\infty} \frac{1}{n^k} = \prod_{p \in \mathbb{P}} \frac{p^k}{p^k - 1} \tag{1}$$

Where p is the n-th prime number and $k \in \mathbf{C}$.

This series is convergent for values of $Re(k) > 1$, however it is divergent for values of $Re(k) \leq 1$.

Euler was able to find a closed formula for even powers, $2k$ when $k \in \mathbb{N}$:

$$\sum_{n=1}^{\infty} \frac{1}{n^{2k}} = \frac{(-1)^{k-1}(2\pi)^{2k} B_{2k}}{2(2k)!} \tag{2}$$

Where B_{2k} are Bernoulli numbers; $B_0 = 1$, $B_1 = -1/2$, $B_2 = 1/6$, etc.

1.2 The Euler-Riemann zeta function $\zeta(k)$.

Riemann introduced the function $\zeta(k)$ [3], making it equal to the series of the sum of the inverse of k-th power inverses in the convergence range $Re(k) > 1$:

$$\zeta(k) = \sum_{n=1}^{\infty} \frac{1}{n^k} \tag{3}$$

And it manages to give continuity to the function, in the range of the complex plane, where the series diverges through the functional equation:

$$\zeta(k) = 2^k \pi^{k-1} sin\left(\frac{\pi k}{2}\right) \Gamma(1-k)\zeta(1-k) \tag{4}$$

Where Γ is the gamma function.

If $Re(k) < 0$, then $\zeta(k)$ can be calculated with the functional equation using of the value of the convergence of the series of the inverse of the powers $\sum_{n=1}^{\infty} \frac{1}{n^{1-k}} = \zeta(1-k)$, so for example for $k = -1$:

Example 1

$$\zeta(-1) = 2^{-1}\pi^{-1-1} sin\left(\frac{\pi(-1)}{2}\right) \Gamma(1-(-1))\zeta(1-(-1))$$
$$\zeta(-1) = 2^{-1}\pi^{-2} sin\left(\frac{\pi(-1)}{2}\right) \Gamma(2))\zeta(2)$$
$$\zeta(-1) = 2^{-1}\pi^{-2}(-1)(1)\frac{\pi^2}{6}$$
$$\zeta(-1) = -\frac{1}{12}$$

From the functional equation, we deduce that for even negative values of k the function $\zeta(k) = 0$, at these *"zeros"* Riemann called *"Trivial zeros"*. There also exist values of k that lie within the range $0 < Re(k) < 1$ that makes the function $\zeta(k) = 0$, these values of k are called *"Nontrivial zeros"* of the function $\zeta(k)$ and which Riemann conjectured all lie on the straight line $Re(k) = \frac{1}{2}$.

The conjecture cannot be proved with the Riemann functional equation alone, because the function is redundant for the so-called critical range: $0 < Re(k) < 1$, for example:

Example 2

$$\zeta(0.1) = 2^{0.1}\pi^{-0.9} sin\left(\frac{\pi * 0.1}{2}\right)\Gamma(0.9)\zeta(0.9)$$

y

$$\zeta(0.9) = 2^{0.9}\pi^{-0.1} sin\left(\frac{\pi * 0.9}{2}\right)\Gamma(0.1)\zeta(0.1)$$

Neither $\zeta(0.1)$*, nor* $\zeta(0.9)$ *can be solved. .*

To calculate the values of $\zeta(k)$ in the critical range $0 < Re(k) < 1$, must to be used numerical methods that calculate approximate values of $\zeta(k)$, which do not prove the hypothesis despite the fact that all computationally obtained non-trivial zeros have the value of $Re(k) = \frac{1}{2}$.

2 Bernoulli numbers and the sum of k-th power.

In mathematics, the Bernoulli numbers B_k is a set of successive rational numbers with relevant importance in number theory. They appear in *Combinatorics*, in the expansion of the tangent functions and the hyperbolic tangent by Taylor series. As we have already seen, Euler obtained a closed formula for $\zeta(k)$ when k is a positive even number. If we replace Euler's formula in the Riemann functional equation, we obtain another closed formula for negative integer values of k:

$$\zeta(-k) = -\frac{B_{k+1}}{k+1} \tag{5}$$

Where $k \in \mathbb{N}$

They are called Bernoulli numbers because Abraham de Moivre named them that way, in honor of Jakob Bernoulli, the first mathematician who studied them. There are several ways to obtain the values of B_k, but they were obtained for the first time by Jakob Bernoulli, using series of sum of k-th power. In general one can obtain the sum of k-th power $S_k(n)$, as a function of: $S_{k-1}(n)$, $S_{k-2}(n)$, $S_{k-3}(n)$,...$S_0(n)$ where $k \in \mathbb{N}$:

$$\sum_{n=1}^{n} n^k = S_k(n) = \frac{1}{k+1}\left[(n+1)^{k+1} - 1 - \sum_{m=0}^{k-1}\binom{k+1}{m}S_m(n)\right] \tag{6}$$

In a posthumous publication by Jakob Bernoulli [1], we can find a listing of the sums of powers up to $k = 10$:

$$\sum_{n=1}^{n} n^k = S_k(n)$$
$$\sum_{n=1}^{n} n^0 = S_0(n) = n$$
$$\sum_{n=1}^{n} n^1 = S_1(n) = \frac{1}{2}n^2 + \frac{1}{2}n$$
$$\sum_{n=1}^{n} n^2 = S_2(n) = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n$$
$$\sum_{n=1}^{n} n^3 = S_3(n) = \frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}n^2$$
$$\sum_{n=1}^{n} n^4 = S_4(n) = \frac{1}{5}n^5 + \frac{1}{2}n^4 + \frac{1}{3}n^3 - \frac{1}{30}n$$
$$\sum_{n=1}^{n} n^5 = S_5(n) = \frac{1}{6}n^6 + \frac{1}{2}n^5 + \frac{5}{12}n^4 - \frac{1}{12}n^2$$
$$\sum_{n=1}^{n} n^6 = S_6(n) = \frac{1}{7}n^7 + \frac{1}{2}n^6 + \frac{1}{2}n^5 - \frac{1}{6}n^3 + \frac{1}{42}n$$
$$\sum_{n=1}^{n} n^7 = S_7(n) = \frac{1}{8}n^8 + \frac{1}{2}n^7 + \frac{7}{12}n^6 - \frac{7}{24}n^4 + \frac{1}{12}n^2$$
$$\sum_{n=1}^{n} n^8 = S_8(n) = \frac{1}{9}n^9 + \frac{1}{2}n^8 + \frac{2}{3}n^7 - \frac{7}{15}n^5 + \frac{2}{9}n^3 - \frac{1}{30}n$$
$$\sum_{n=1}^{n} n^9 = S_9(n) = \frac{1}{10}n^{10} + \frac{1}{2}n^9 + \frac{3}{4}n^8 - \frac{7}{10}n^6 + \frac{1}{2}n^4 - \frac{3}{20}n^2$$
$$\sum_{n=1}^{n} n^{10} = S_{10}(n) = \frac{1}{11}n^{11} + \frac{1}{2}n^{10} + \frac{5}{6}n^9 - n^7 + n^5 - \frac{1}{2}n^3 + \frac{5}{66}n$$
$$\sum_{n=1}^{n} n^{11} = S_{11}(n) = \frac{1}{12}n^{12} + \frac{1}{2}n^{11} + \frac{11}{12}n^{10} - \frac{11}{8}n^8 + \frac{11}{6}n^6 - \frac{11}{8}n^4 + \frac{5}{12}n^2$$

2.1 Obtaining the Bernoulli numbers by the sum of powers.

It was known that to obtain the Bernoulli numbers it was necessary to derive $S_k(n)$ and evaluate it at zero, however this concept is not entirely correct since for B_1 when applying this concept it is not possible to obtain the value of $B_1 = -1/2$, value obtained by other methods. The correct way to obtain the Bernoulli numbers by the sum of k-th power is with the equation:

$$B_k = (-1)^k S_k'(0) \tag{7}$$

The equation (7) *will be proofed later on*

For example, to obtain B_2:

Example 3

$$B_2 = (-1)^2 S_2'(0)$$
$$B_2 = (1)\left[\frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n\right]'_{n=0}$$
$$B_2 = \left[n^2 + n + \frac{1}{6}\right]_{n=0}$$
$$B_2 = \frac{1}{6}$$

Similarly, from the function $S_k(n)$ all Bernoulli numbers are obtained: $B_0 = 1, B_1 = -1/2, B_2 = 1/6, B_3 = 0, B_4 = -1/30, B_5 = 0, B_6 = 1/42, B_7 = 0, B_8 = -1/30, B_9 = 0, B_{10} = 5/66, B_{11} = 0$.

It is noted that:

$$B_k = 0 \ / \ k = 2m+1, m \in \mathbb{N}$$

2.2 Simplified formula to find $S_k(n)$.

Another way to write the formula for the sum of k-th power is as follows:

$$S_k(n) = \sum_{p=1}^{1+k} A_p(k) n^p \tag{8}$$

Where

$$A_p(k) = \frac{(-1)^{1+k-p}}{1+k}\binom{1+k}{p} B_{1+k-p} \tag{9}$$

And B_k is obtained by:

$$B_k = -\frac{1}{1+k}\sum_{m=0}^{k-1}\binom{1+k}{m} B_m \tag{10}$$

Equation (8) can be rewritten without using combinatorics in order to allow the variable k to be a real number or even a complex number, by following the steps below:

Factorizing $1+k$ and developing the summation and binomial coefficient of $S_k(n)$:

$$S_k(n) = \frac{1}{1+k}\left[\frac{(-1)^k(k+1)!}{1!k!}B_k n + \frac{(-1)^{k-1}(k+1)!}{2!(k-1)!}B_{k-1}n^2 + \frac{(-1)^{k-2}(k+1)!}{3!(k-2)!}B_{k-2}n^3 + \right.$$
$$\left. \frac{(-1)^{k-3}(k+1)!}{4!(k-3)!}B_{k-3}n^4 + \cdots + \frac{(-1)^1(k+1)!}{k!1!}B_1 n^k + \frac{(-1)^0(k+1)!}{(k+1)!(0)!}B_0 n^{k+1}\right]$$

Rearranging terms and accommodating the factorials in order to simplify:

$$S_k(n) = \frac{1}{1+k}\left[\frac{(-1)^0(k+1)!}{(k+1)!0!}B_0 n^{k+1} + \frac{(-1)^1(k+1)!}{k!1!}B_1 n^k + \frac{(-1)^2(k+1)!}{(k-1)!2!}B_2 n^{k-1} + \cdots \right.$$
$$\left. \cdots + \frac{(-1)^k(k+1)!}{1!k!}B_k n\right]$$

$$S_k(n) = \frac{1}{1+k}\left[\frac{(-1)^0(k+1)!}{(k+1)!0!}B_0 n^{k+1} + \frac{(-1)^1 k!(k+1)}{k!1!}B_1 n^k + \frac{(-1)^2(k-1)!k(k+1)}{(k-1)!2!}B_2 n^{k-1} + \cdots \right.$$
$$\left. \cdots + \frac{(-1)^k(k+1)!}{1!k!}B_k n\right]$$

$$S_k(n) = \frac{1}{1+k}\left[\frac{(-1)^0}{0!}B_0 n^{k+1} + \frac{(-1)^1(k+1)}{1!}B_1 n^k + \frac{(-1)^2 k(k+1)}{2!}B_2 n^{k-1} + \cdots \right.$$
$$\left. \cdots + \frac{(-1)^k(k+1)!}{1!k!}B_k n\right]$$

Rewriting as a summation of a product of factors:

$$S_k(n) = \sum_{p=1}^{1+k} \frac{(-1)^{p-1} B_{p-1}}{(1+k)(p-1)!} \prod_{m=1}^{p-1} (2+k-m) n^{2+k-p} \tag{11}$$

Note 1 *The product* $\prod_{m=1}^{p-1}(2+k-m) = 1 : p = 1$

It can also be expressed as the sum of higher order derivatives:

$$S_k(n) = \sum_{p=1}^{1+k} \frac{(-1)^{p-1} B_{p-1}}{(1+k)(p-1)!} * \frac{d^{p-1}}{dn^{p-1}} n^{2+k-p} \tag{12}$$

where:

$$C_p(k) = \frac{(-1)^{p-1} B_{p-1}}{(1+k)(p-1)!} \prod_{m=1}^{p-1} (2+k-m) \tag{13}$$

With the equation (11) it is possible to prove equation (7)

Proof of equation (7):

Deriving the equation (11) and evaluating at zero we obtain:

$$S_k'(0) = \sum_{p=1}^{1+k} (2+k-p) \frac{(-1)^{p-1} B_{p-1}}{(1+k)(p-1)!} \prod_{m=1}^{p-1} (2+k-m)(0)^{1+k-p}$$

Where the only term different from zero is when $p = k+1$, so the expression reduces to:

$$S_k'(0) = \frac{(-1)^k (k+1)!}{(1+k)k!} B_k$$

$$S_k'(0) = \frac{(-1)^k (k+1)!}{(k+1)!} B_k$$

$$S_k'(0) = (-1)^k B_k$$

Reordering:

$$B_k = (-1)^k S_k'(0)$$

□.

3 The funtion $S_k^*(n)$.

It is now possible to propose a function $S_k^*(n)$, which is the extension of the function $S_k(n)$ but this time instead of adding $k+1$ terms, the sum of terms will be infinite.

Formula to calculate the sum of k-th power:

$$S_k(n) = \sum_{p=1}^{1+k} \frac{(-1)^{p-1} B_{p-1}}{(1+k)(p-1)!} \prod_{m=1}^{p-1} (2+k-m) n^{2+k-p}$$

Proposed function:

$$S_k^*(n) = \sum_{p=1}^{\infty} \frac{(-1)^{p-1} B_{p-1}}{(1+k)(p-1)!} \prod_{m=1}^{p-1} (2+k-m) n^{2+k-p} \tag{14}$$

Note 2 *We will use the symbol* * *to distinguish the sum of powers* $S_k(n)$*, from the proposed function* $S_k^*(n)$*.*

Now let's define $\Delta_{-k}(n)$ as the difference between the two functions:

$$\Delta_{-k}(n) = \sum_{n=1}^{n} n^k - S_k^*(n) \tag{15}$$

And Δ_{-k} as the limit when $n \to \infty$ of $\Delta_{-k}(n)$:

$$\Delta_{-k} = \lim_{n\to\infty} \left[\sum_{n=1}^{n} n^k - S_k^*(n) \right] \tag{16}$$

Replacing equation (13) in equation(16) we obtain:

$$\Delta_{-k} = \lim_{n\to\infty} \left[\sum_{n=1}^{n} n^k - \sum_{p=1}^{\infty} \frac{(-1)^{p-1} B_{p-1}}{(1+k)(p-1)!} \prod_{m=1}^{p-1} (2+k-m) n^{2+k-p} \right] \tag{17}$$

Changing the variable -k by k, we obtain:

$$\Delta_{k} = \lim_{n\to\infty} \left[\sum_{n=1}^{n} \frac{1}{n^k} - S_{-k}^*(n) \right] \tag{18}$$

$$\Delta_{k} = \lim_{n\to\infty} \left[\sum_{n=1}^{n} \frac{1}{n^k} - \sum_{p=1}^{\infty} \frac{(-1)^{p-1} B_{p-1}}{(1-k)(p-1)!} \prod_{m=1}^{p-1} (2-k-m) n^{2-k-p} \right] \tag{19}$$

3.1 Verification of the convergence of the function Δ_k.

To verify convergence we will consider 3 cases, where $k \in \mathbb{C}$:

Case 1. Let $Re(k) > 1$:

As $Re(k) > 1$ and $p \geq 1$, it follows that all exponent of n of $S_{-k}(n)$ is:

$$Re(2-k-p) < 0$$

Therefore: when $n \to \infty$ and $k > 1$ it is satisfied that the value of $n^{2-k-p} = 0$. Replacing n^{2-k-p} in equation (19) gives:

$$\Delta_{k} = \lim_{n\to\infty} \left[\sum_{n=1}^{n} \frac{1}{n^k} - C_1 * 0 - C_2 * 0 - C_3 * 0 - \cdots - C_p * 0 \right]$$

$$\Delta_{k} = \left[\sum_{n=1}^{\infty} \frac{1}{n^k} - 0 - 0 - 0 - \cdots \right]$$

$$\Delta_{k} = \sum_{n=1}^{\infty} \frac{1}{n^k} \iff Re(k) > 1 \tag{20}$$

Therefore it is concluded that: when $Re(k) > 1$ the value of Δ_k converges, and is equal to the sum of k-th power inverses, and is equal to $\zeta(k)$.

Case 2. Let $k = 1$:

$$\Delta_{1} = \lim_{n\to\infty} \left[\sum_{n=1}^{\infty} \frac{1}{n^k} - C_1 n^0 - C_2 n^{-1} - C_2 n^{-2} - \cdots \right]$$

$$\Delta_{1} = \left[\sum_{n=1}^{\infty} \frac{1}{n^k} - C_1 - 0 - 0 - \cdots \right]$$

Since the sum of powers when $k = -1$ is infinite then:

$$\Delta_1 = \left[\infty - \frac{(-1)^{-1+1}}{(-1+1)(-1+1)!} B_{-1+1} - 0 - 0 - \cdots \right]$$

$$\Delta_1 = \left[\infty - \frac{(-1)^0}{(0)(0)!} B_0 \right]$$

$$\Delta_1 = undetermined \tag{21}$$

We conclude that: When $k = 1$ Δ_k is undetermined, and is equal to $\zeta(k)$.

Case 3. Let $Re(k) < 1$:

It is equivalent to consider Re(k)¿1 in equation (17).
Let us first separate the summation of $C_p(k)n^{k+2-p}$ into two, where the first rank of p will be from 1 to $(k+1)$ and the second rank, from $(k+2)$ to (∞):

$$\Delta_{-k} = \lim_{n\to\infty} \left[\sum_{n=1}^{n} n^k - \sum_{p=1}^{k+1} C_p(k)n^{k+2-p} - \sum_{p=k+2}^{\infty} C_p(k)n^{k+2-p} \right]$$

Since:

$$\sum_{n=1}^{n} n^k = \sum_{p=1}^{k+1} C_p(k)n^{k+2-p}$$

We obtain:

$$\Delta_{-k} = -\lim_{n\to\infty} \left[\sum_{p=k+2}^{\infty} C_p(k)n^{k+2-p} \right]$$

Developing the summation:

$$\Delta_{-k} = -\lim_{n\to\infty} \left[C_{k+2}(k)n^0 + C_{k+3}(k)n^{-1} + C_{k+4}(k)n^{-2} + \cdots \right]$$

Replacing the limit:

$$\Delta_{-k} = -[C_{k+2}(k) + C_{k+3}(k)0 + C_{k+4}(k)0 + \cdots]$$

Simplifying:

$$\Delta_{-k} = -C_{k+2}(k) \quad para \quad k < 0 \tag{22}$$

Replacing the value of $C_{k+2}(k)$ gives:

$$\Delta_{-k} = -\frac{(-1)^{k+1}B_{k+1}}{(k+1)(k+1)!} \prod_{m=1}^{k+1} (k+2-m) \tag{23}$$

Solving the product:

$$\Delta_{-k} = -\frac{(-1)^{k+1}B_{k+1}}{(k+1)(k+1)!} [(k+1) * k * (k-1) * (k-2) * \cdots * 1]$$

$$\Delta_{-k} = -\frac{(-1)^{k+1}B_{k+1}}{(k+1)(k+1)!}(k+1)!$$

Simplifying terms:

$$\Delta_{-k} = \frac{(-1)^k B_{k+1}}{(k+1)} \tag{24}$$

We conclude that: Equation converges, and this equation is equivalent to the well-known formula for the function $\zeta(-k)$ equation (5), and also satisfies when $Re(k) = 0$.

Theorem 1 *Let $S^*_{-k}(n)$ be defined as:*

$$S^*_{-k}(n) = \sum_{p=1}^{\infty} \frac{(-1)^{p-1}B_{p-1}}{(1-k)(p-1)!} \prod_{m=1}^{p-1} (2-k-m)n^{2-k-p} \tag{25}$$

And the function Δ_k:

$$\Delta_k = \lim_{n\to\infty} \left[\sum_{n=1}^{n} \frac{1}{n^k} - S^*_{-k}(n) \right]$$

It can be written:

$$\Delta_k = \lim_{n\to\infty} \left[\sum_{n=1}^{n} \frac{1}{n^k} - \sum_{p=1}^{\infty} \frac{(-1)^{p-1}B_{p-1}}{(1-k)(p-1)!} \prod_{m=1}^{p-1} (2-k-m)n^{2-k-p} \right] \tag{26}$$

Since the limit when $n \to \infty$ of the function converges for all $k \in \mathbb{C} - \{0\}$, moreover its value coincides with that of the function $\zeta(-k)$ therefore one can define the function $\zeta(k)$ as follows:

$$\zeta(k) = \lim_{n\to\infty} \left[\sum_{n=1}^{n} \frac{1}{n^k} - S^*_{-k}(n) \right] \tag{27}$$

Where $S^*_{-k}(n)$ is the related function of $\sum_{n=1}^{n} \frac{1}{n^k}$.

Example 4 *Now let's calculate a example, when $k \in \mathbb{C}$*

Let $k = -0.4 - 7i$:

We will tabulate the results and compare with $\zeta(0.4 + 7i)$:

n	$\sum_{n=1}^{n} n^{-0.4-7i}$	$S^*_{-0.4-7i}(n)$	$\Delta_{0.4+7i}$
10	$0.57811497 + 0.019131365i$	$-0.441562086 - 0.398323033i$	$1.019677056 + 0.417454399i$
100	$2.847975959 + 1.750845166i$	$1.82847357 + 1.333500916i$	$1.019502389 + 0.417344249i$
1000	$36.34167952 - 5.123159998i$	$35.32217705 - 5.540504261i$	$1.01950247 + 0.4173344263i$

If we calculate the value of $\zeta(0.4+7i)$ by some numerical method we obtain the approximate value of $1.01950247 + 0.417344263i$, and comparing with the value of $\Delta_{0.4+7i}$ we verify that $\Delta_{0.4+7i} = \zeta(0.4 + 7i)$.

4 Proof of the Riemann Hypothesis.

Let k be a complex number such that $k = a + bi$ where $a, b \in \mathbb{R}$, $\zeta(k) = 0$.

From the Riemann functional equation:

$$\zeta(a+bi) = 2^{a+bi}\pi^{a+bi-1} \sin\left(\frac{\pi(a+bi)}{2}\right) \Gamma(1-a-bi)\zeta(1-a-bi)$$

It is known that for the critical band range $0 < a < 1$, the terms:

$$2^{a+bi}\pi^{a+bi-1} \sin\left(\frac{\pi(a+bi)}{2}\right) \Gamma(1-a-bi) \neq 0$$

Therefore, including its conjugates, it must comply:

$$\zeta(a+bi) = \zeta(1-a-bi) = \zeta(a-bi) = \zeta(1-a+bi) = 0 \tag{28}$$

On the other hand, we can write the equation of zeta (27) within the critical band, when $0 < a < 1$

$$\zeta(a+bi) = \lim_{n\to\infty} \left[\sum_{n=1}^{n} \frac{1}{n^{a+bi}} - \sum_{p=1}^{\infty} \frac{(-1)^{p-1}B_{p-1}}{(1-a-bi)(p-1)!} \prod_{m=1}^{p-1} (2-a-bi-m)n^{2-a-bi-p} \right]$$

$$\zeta(a+bi) = \lim_{n\to\infty}\left[\sum_{n=1}^{n}\frac{1}{n^{a+bi}} - \frac{(-1)^0B_0}{(1-a-bi)(0)!}n^{1-a-bi} - \frac{(-1)^1B_1}{(1-a-bi)(1)!}n^{-a-bi} - \frac{(-1)^2B_2}{(1-a-bi)(2)!}n^{-1-a-bi}\cdots\right]$$

As $0 < a < 1$ and $n \to \infty$ the equation simplifies to:

$$\zeta(a+bi) = \lim_{n\to\infty}\left[\sum_{n=1}^{n}\frac{1}{n^{a+bi}} - \frac{(-1)^0B_0}{(1-a-bi)(0)!}n^{1-a-bi}\right]$$

Substituting B_0 and solving:

$$\zeta(a+bi) = \lim_{n\to\infty}\left[\sum_{n=1}^{n}\frac{1}{n^{a+bi}} - \frac{1}{(1-a-bi)}n^{1-a-bi}\right] \tag{29}$$

Applying properties of complex numbers and bringing to polar form:

$$\zeta(a+bi) = \lim_{n\to\infty}\left[\sum_{n=1}^{n}n^{-a}e^{-ib\ln n} - \frac{n^{1-a}}{\sqrt{(1-a)^2+b^2}}e^{i\arctan\left(\frac{b}{a-1}\right)}e^{-iblnn}\right]$$

As $\zeta(a+bi) = 0$ and then applying euler identities for sines and cosines:

$$0 = \lim_{n\to\infty}\left[\sum_{n=1}^{n}n^{-a}e^{-ib\ln n} - \frac{n^{1-a}}{\sqrt{(1-a)^2+b^2}}e^{i\left[\arctan\left(\frac{b}{a-1}\right)-blnn\right]}\right]$$

$$\sum_{n=1}^{\infty}n^{-a}e^{-ib\ln n} = \lim_{n\to\infty}\left[\frac{n^{1-a}}{\sqrt{(1-a)^2+b^2}}\left[\cos\left(\arctan\left(\frac{b}{a-1}\right)-blnn\right)+i\sin\left(\arctan\left(\frac{b}{a-1}\right)-blnn\right)\right]\right] \tag{30}$$

Where the modulus and argument of $S^*_{-a-bi}(n)$ are:

$$\left\|S^*_{-a-bi}(n)\right\| = \frac{n^{1-a}}{\sqrt{(1-a)^2+b^2}} \tag{31}$$

$$\arg(S^*_{-a-bi}) = \arctan\left(\frac{b}{a-1}\right) - blnn \tag{32}$$

Then by Euler's identity and equating the modules, we have:

$$\sqrt{\left[\sum_{n=1}^{\infty}n^{-a}\cos(-b\ln n)\right]^2 + \left[\sum_{n=1}^{\infty}n^{-a}\sin(-b\ln n)\right]^2} = \lim_{n\to\infty}\left[\frac{n^{1-a}}{\sqrt{(1-a)^2+b^2}}\right] \tag{33}$$

Extending the series within the square root we obtain:

$$\begin{gathered}
\left[\sum_{n=1}^{\infty}n^{-a}\cos(-b\ln n)\right]^2 + \left[\sum_{n=1}^{\infty}n^{-a}\sin(-b\ln n)\right]^2 = \\
\lim_{n\to\infty}\left[1^{-2a}\cos^2(-b\ln 1) + 2^{-2a}\cos^2(-b\ln 2) + 3^{-2a}\cos^2(-b\ln 3) + \cdots + n^{-2a}\cos^2(-b\ln n)\right. \\
+2\{1^{-a}2^{-a}\cos(-b\ln 1)\cos(-b\ln 2) + 1^{-a}3^{-a}\cos(-b\ln 1)\cos(-b\ln 3) + \cdots + 1^{-a}n^{-a}\cos(-b\ln 1)\cos(-b\ln n)\} \\
+2\{2^{-a}3^{-a}\cos(-b\ln 2)\cos(-b\ln 3) + 2^{-a}4^{-a}\cos(-b\ln 2)\cos(-b\ln 4) + \cdots + 2^{-a}n^{-a}\cos(-b\ln 2)\cos(-b\ln n)\} \\
+2\{3^{-a}4^{-a}\cos(-b\ln 3)\cos(-b\ln 4) + 3^{-a}5^{-a}\cos(-b\ln 3)\cos(-b\ln 5) + \cdots + 3^{-a}n^{-a}\cos(-b\ln 3)\cos(-b\ln n)\} \\
+\cdots+\cdots+2\{(n-1)^{-a}n^{-a}\cos(-b\ln(n-1))\cos(-b\ln n)\} \\
+1^{-2a}\sin^2(-b\ln 1) + 2^{-2a}\sin^2(-b\ln 2) + 3^{-2a}\sin^2(-b\ln 3) + \cdots + n^{-2a}\sin^2(-b\ln n) \\
+2\{1^{-a}2^{-a}\sin(-b\ln 1)\sin(-b\ln 2) + 1^{-a}3^{-a}\sin(-b\ln 1)\sin(-b\ln 3) + \cdots + 1^{-a}n^{-a}\sin(-b\ln 1)\sin(-b\ln n)\} \\
+2\{2^{-a}3^{-a}\sin(-b\ln 2)\sin(-b\ln 3) + 2^{-a}4^{-a}\sin(-b\ln 2)\sin(-b\ln 4) + \cdots + 2^{-a}n^{-a}\sin(-b\ln 2)\sin(-b\ln n)\} \\
+2\{3^{-a}4^{-a}\sin(-b\ln 3)\sin(-b\ln 4) + 3^{-a}5^{-a}\sin(-b\ln 3)\sin(-b\ln 5) + \cdots + 3^{-a}n^{-a}\sin(-b\ln 3)\sin(-b\ln n)\} \\
\left.+\cdots+\cdots+2\{(n-1)^{-a}n^{-a}\sin(-b\ln(n-1))\sin(-b\ln n)\}\right]
\end{gathered}$$

Applying trigonometric idenities:

$$\left[\sum_{n=1}^{\infty} n^{-a}\cos(-b\ln n)\right]^2 + \left[\sum_{n=1}^{\infty} n^{-a}\sin(-b\ln n)\right]^2 = \lim_{n\to\infty}\left[1^{-2a} + 2^{-2a} + 3^{-2a} + \cdots + n^{-2a}\right.$$
$$+2\{1^{-a}2^{-a}\cos(b\ln 1 - b\ln 2) + 1^{-a}3^{-a}\cos(b\ln 1 - b\ln 3) + \cdots + 1^{-a}n^{-a}\cos(b\ln 1 - b\ln n)\}$$
$$+2\{2^{-a}3^{-a}\cos(b\ln 2 - b\ln 3) + 2^{-a}4^{-a}\cos(b\ln 2 - b\ln 4) + \cdots + 2^{-a}n^{-a}\cos(b\ln 2 - b\ln n)\}$$
$$+2\{3^{-a}4^{-a}\cos(b\ln 3 - b\ln 4) + 3^{-a}5^{-a}\cos(b\ln 3 - b\ln 5) + \cdots + 3^{-a}n^{-a}\cos(b\ln 3 - b\ln n)\}$$
$$\left. + \cdots + \cdots + 2\{(n-1)^{-a}n^{-a}\cos(b\ln(n-1) - b\ln n)\}\right]$$

Rewriting the series, we obtain:

$$\left[\sum_{n=1}^{\infty} n^{-a}\cos(-b\ln n)\right]^2 + \left[\sum_{n=1}^{\infty} n^{-a}\sin(-b\ln n)\right]^2 = \lim_{n\to\infty}\left[\sum_{m=1}^{n}\frac{1}{m^{2a}} + 2\sum_{m=1}^{n-1}\sum_{t=1}^{m}(m+1)^{-a}t^{-a}\cos\left(\ln\left(\frac{m+1}{t}\right)\right)\right] \tag{34}$$

Replacing (34) in (33) we get:

$$\lim_{n\to\infty}\sqrt{\sum_{m=1}^{n}\frac{1}{m^{2a}} + 2\sum_{m=1}^{n-1}\sum_{t=1}^{m}(m+1)^{-a}t^{-a}\cos\left(\ln\left(\frac{m+1}{t}\right)\right)} = \lim_{n\to\infty}\left[\frac{n^{1-a}}{\sqrt{(1-a)^2+b^2}}\right] \tag{35}$$

We know that the range for a is $0 < a < 1$, and $n \to \infty$, so the right hand side of equation (35) tends to infinity, :

$$\lim_{n\to\infty}\sqrt{\sum_{m=1}^{n}\frac{1}{m^{2a}} + 2\sum_{m=1}^{n-1}\sum_{t=1}^{m}(m+1)^{-a}t^{-a}\cos\left(\ln\left(\frac{m+1}{t}\right)\right)} = \infty \tag{36}$$

If we add to equation (36) the modulus of the next term to the series, we obtain:

$$\lim_{n\to\infty}\left\{\left[\sqrt{\sum_{m=1}^{n}\frac{1}{m^{2a}} + 2\sum_{m=1}^{n-1}\sum_{t=1}^{m}(m+1)^{-a}t^{-a}\cos\left(\ln\left(\frac{m+1}{t}\right)\right)}\right]^2 + \left[\frac{1}{(n+1)^a}\right]^2\right.$$
$$\left. + \left[2\sqrt{\sum_{m=1}^{n}\frac{1}{m^{2a}} + 2\sum_{m=1}^{n-1}\sum_{t=1}^{m}(m+1)^{-a}t^{-a}\cos\left(\ln\left(\frac{m+1}{t}\right)\right)}\right]\left[\frac{1}{(n+1)^a}\right]\cos\phi\right\} = \infty \tag{37}$$

where ϕ is the angle between the cumulative modulus of the summation and the (n+1)-th term.
From equation (32) we know that ϕ is:

$$\phi = \arctan\left(\frac{b}{1-a}\right) \tag{38}$$

Moreover, we know that $\zeta(k)$ has infinite nontrivial zeros, so there is a b as large as infinity, therefore:

$$\phi = \lim_{b\to\infty}\arctan\left(\frac{b}{1-a}\right) = \frac{\pi}{2} \tag{39}$$

Replacing (39) in (37):

$$\lim_{n\to\infty}\left\{\left[\sqrt{\sum_{m=1}^{n}\frac{1}{m^{2a}} + 2\sum_{m=1}^{n-1}\sum_{t=1}^{m}(m+1)^{-a}t^{-a}\cos\left(\ln\left(\frac{m+1}{t}\right)\right)}\right]^2 + \left[\frac{1}{(n+1)^a}\right]^2\right.$$
$$\left. + \left[2\sqrt{\sum_{m=1}^{n}\frac{1}{m^{2a}} + 2\sum_{m=1}^{n-1}\sum_{t=1}^{m}(m+1)^{-a}t^{-a}\cos\left(\ln\left(\frac{m+1}{t}\right)\right)}\right]\left[\frac{1}{(n+1)^a}\right]\cos\left(\frac{\pi}{2}\right)\right\} = \infty \tag{40}$$

Since $\cos(\pi/2) = 0$, equation (40) reduces to:

$$\lim_{n\to\infty}\left\{\left[\sqrt{\sum_{m=1}^{n}\frac{1}{m^{2a}} + 2\sum_{m=1}^{n-1}\sum_{t=1}^{m}(m+1)^{-a}t^{-a}\cos\left(\ln\left(\frac{m+1}{t}\right)\right)}\right]^2 + \left[\frac{1}{(n+1)^a}\right]^2\right\} = \infty \quad (41)$$

$$\lim_{n\to\infty}\left\{\sum_{m=1}^{n}\frac{1}{m^{2a}} + 2\sum_{m=1}^{n-1}\sum_{t=1}^{m}(m+1)^{-a}t^{-a}\cos\left(\ln\left(\frac{m+1}{t}\right)\right) + \frac{1}{(n+1)^{2a}}\right\} = \infty \quad (42)$$

Note that when adding the (n+1)-th term, only the first summation increases and the second summation remains constant:

$$\lim_{n\to\infty}\left\{\sum_{m=1}^{n+1}\frac{1}{m^{2a}} + 2\sum_{m=1}^{n-1}\sum_{t=1}^{m}(m+1)^{-a}t^{-a}\cos\left(\ln\left(\frac{m+1}{t}\right)\right)\right\} = \infty \quad (43)$$

Therefore, it must be fulfilled that:

$$\lim_{n\to\infty}\sum_{m=1}^{n+1}\frac{1}{m^{2a}} = \infty \quad (44)$$

The series of equation (44) converges when $a > 1/2$, therefore it must be satisfied that:

$$\lim_{n\to\infty}\sum_{m=1}^{n+1}\frac{1}{m^{2a}} = \infty \iff a \leq \frac{1}{2} \quad (45)$$

Then equation (36) is only true for values of $a \leq 1/2$: Consequently, we can state that every non-trivial zero of the zeta function has real part less than or equal to 1/2:

$$\exists\, b : \zeta(a+bi) = 0 \iff a \leq \frac{1}{2} \quad (46)$$

From equation (28) and (46), it must be fulfilled:

$$\left.\begin{array}{r} a \leq \frac{1}{2} \\ 1-a \leq \frac{1}{2} \end{array}\right\} \quad (47)$$

Whose solution is:

$$a = \frac{1}{2} \quad (48)$$

Finally, we can affirm that:

$$\exists\, b : \zeta(a+bi) = 0 \iff a = \frac{1}{2} \quad (49)$$

□.

5 Conclusion

Based on the above, we can state the following theorem:

Theorem 2 *The nontrivial zeros of the Riemann zeta function $\zeta(k)$ have real part equal to 1/2.*

References

[1] Jakob Bernoulli. *Ars conjectandi: opus posthumum: accedit Tractatus de seriebus infinitis; et Epistola gallice scripta de ludo pilae reticularis.* Impensis Thurnisiorum, 1713.

[2] Leonhard Euler. Variae observationes circa series infinitas. *Commentarii academiae scientiarum imperialis Petropolitanae*, 9(1737):160–188, 1737.

[3] Bernhard Riemann. Ueber die anzahl der primzahlen unter einer gegebenen grosse. *Ges. Math. Werke und Wissenschaftlicher Nachlaß*, 2(145-155):2, 1859.

Bernoulli sums of powers and proof that Riemann Hypothesis is true

By:

Eng. Marco Rolando Burgos Chambi

Jun 29th 2022

Dedication.

I thank God for allowing me to finish this paper.
This work is dedicated to all my family who supported me at all times and in the most difficult moments of my life, especially my beloved wife Araceli, my two beautiful children Ocram and Arelys, and my parents who never lost faith in me.