One formula that produces primes of the form 2n+1

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Abstract

Here I present one formula that produces prime numbers.

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Property

Let $\sigma(n)$ denotes the divisor function which sums the divisors of n, an integer ≥ 1. We introduce the function f such that:

$$f(n) = 1 + (n!)^2 - \sigma(n!)(n!)^2 + 2 \sum_{k=1}^{-1+\sigma(n!)} \left\lfloor \frac{k(1+(n!)^2)}{\sigma(n!)} \right\rfloor$$

When $f(n) = 2n + 1$, is $2n + 1$ always prime?

And for example for $n = 6$ we have the prime number 13 that is of the form $2(6) + 1$.

The first examples are given by the following sequence:

Sequence

1, 1, 1, 1, 1, 13, 1, 17, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 61, 1, 1, 1, 1, 1, 1, 1, 1, 61, 1, 1, 1, 193, 1, 1, 1, 757, 61, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 109, 1, 1, 1, 181, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 113...

Proof

Carl Schildkraut proved this property [1].

Let $\{x\} = x - \lfloor x \rfloor$, let $m = n!$, and let $t = \sigma(m)$. Then

$$2\sum_{k=1}^{t-1} \left\lfloor \frac{k(1+m^2)}{t} \right\rfloor = \sum_{k=0}^{t-1} \frac{2k(1+m^2)}{t} - 2\sum_{k=0}^{t-1} \left\{ \frac{k(1+m^2)}{t} \right\};$$

the first sum is $(1+m^2)(t-1)$, and so

$$\begin{aligned} f(n) &= 1 + m^2 - m^2 t + (1+m^2)(t-1) - 2\sum_{k=0}^{t-1} \left\{ \frac{k(1+m^2)}{t} \right\} \\ &= t - 2\sum_{k=0}^{t-1} \left\{ \frac{k(1+m^2)}{t} \right\}. \end{aligned}$$

Let $u = \gcd(1+m^2, t)$. The t values $\{0, 1+m^2, 2(1+m^2), \ldots, (t-1)(1+m^2)\}$ modulo t consist of u copies of each multiple of u in $[0, t)$, and so

$$\sum_{k=0}^{t-1} \left\{ \frac{k(1+m^2)}{t} \right\} = u \sum_{j=0}^{\frac{t}{u}-1} \frac{uj}{t} = \frac{t-u}{2}.$$

This means

$$f(n) = t - 2\frac{t-u}{2} = \gcd(1+(n!)^2, \sigma(n!)).$$

(In particular, if $1+(n!)^2$ and $\sigma(n!)$ are coprime, $f(n) = 1$.)

With this knowledge about $f(n)$, we can tackle the problem at hand. If $f(n) = 2n+1$, then, in particular, $2n+1$ divides $1+(n!)^2$. So, $2n+1$ is relatively prime to $n!$. This means that $2n+1$ cannot have any factors in the set $\{2, \ldots, n\}$. However, every number in $\{n+1, \ldots, 2n\}$ is too large to be a factor of $2n+1$. So, $2n+1$ cannot have any factors strictly between 1 and $2n+1$, and must be prime.

Reference

[1] Carl Schildkraut (https://math.stackexchange.com/users/253966/carl-schildkraut), Primes of the form $2n+1$, URL (version: 2022-06-26): https://math.stackexchange.com/q/4480961