Proof of Riemann hypothesis

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Abstract. This paper is a trial to prove Riemann hypothesis according to the following process. 1. We make one identity regarding x from one equation that gives Riemann zeta function $\zeta(s)$ analytic continuation and 2 formulas ($1/2+a\pm bi$, $1/2-a\pm bi$) that show non-trivial zero point of $\zeta(s)$. 2. We find that the above identity holds only at $a=0$. 3. Therefore non-trivial zero points of $\zeta(s)$ must be $1/2\pm bi$ because a cannot have any value but zero.

1. Introduction

The following (1) gives Riemann zeta function $\zeta(s)$ analytic continuation to $0 < Re(s)$. "$+\cdots\cdots$" means infinite series in all equations in this paper.

$$1 - 2^{-s} + 3^{-s} - 4^{-s} + 5^{-s} - 6^{-s} + \cdots\cdots = (1 - 2^{1-s})\zeta(s) \tag{1}$$

The following (2) shows the zero point of the left side of (1) and also non-trivial zero point of $\zeta(s)$. i is $\sqrt{-1}$.

$$S_0 = 1/2 + a \pm bi \tag{2}$$

The following (3) also shows non-trivial zero point of $\zeta(s)$ by the functional equation of $\zeta(s)$.

$$S_1 = 1 - S_0 = 1/2 - a \mp bi \tag{3}$$

We define the range of a and b as $0 \le a < 1/2$ and $14 < b$ respectively. Then we can show all non-trivial zero points of $\zeta(s)$ by the above (2) and (3). Because non-trivial zero points of $\zeta(s)$ exist in the critical strip of $\zeta(s)$ ($0 < Re(s) < 1$) and non-trivial zero points of $\zeta(s)$ found until now exist in the range of $14 < b$.
We have the following (4) and (5) by substituting S_0 for s in the left side of (1) and putting both the real part and the imaginary part of the left side of (1) at zero respectively.

$$1 = \frac{\cos(b\log 2)}{2^{1/2+a}} - \frac{\cos(b\log 3)}{3^{1/2+a}} + \frac{\cos(b\log 4)}{4^{1/2+a}} - \frac{\cos(b\log 5)}{5^{1/2+a}} + \cdots\cdots \tag{4}$$

$$0 = \frac{\sin(b\log 2)}{2^{1/2+a}} - \frac{\sin(b\log 3)}{3^{1/2+a}} + \frac{\sin(b\log 4)}{4^{1/2+a}} - \frac{\sin(b\log 5)}{5^{1/2+a}} + \cdots\cdots \tag{5}$$

We also have the following (6) and (7) by substituting S_1 for s in the left side of (1) and putting both the real part and the imaginary part of the left side of (1) at zero

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respectively.

$$1 = \frac{\cos(b\log 2)}{2^{1/2-a}} - \frac{\cos(b\log 3)}{3^{1/2-a}} + \frac{\cos(b\log 4)}{4^{1/2-a}} - \frac{\cos(b\log 5)}{5^{1/2-a}} + \cdots\cdots \tag{6}$$

$$0 = \frac{\sin(b\log 2)}{2^{1/2-a}} - \frac{\sin(b\log 3)}{3^{1/2-a}} + \frac{\sin(b\log 4)}{4^{1/2-a}} - \frac{\sin(b\log 5)}{5^{1/2-a}} + \cdots\cdots \tag{7}$$

2. The identity regarding x

We define $f(n)$ as follows.

$$f(n) = \frac{1}{n^{1/2-a}} - \frac{1}{n^{1/2+a}} \geq 0 \qquad (n = 2, 3, 4, 5, \cdots\cdots) \tag{8}$$

We have the following (9) from the above (4) and (6) with the method shown in item 1.1 of [Appendix 1: Equation construction].

$$0 = f(2)\cos(b\log 2) - f(3)\cos(b\log 3) + f(4)\cos(b\log 4) - f(5)\cos(b\log 5) + \cdots\cdots \tag{9}$$

We also have the following (10) from the above (5) and (7) with the method shown in item 1.2 of [Appendix 1].

$$0 = f(2)\sin(b\log 2) - f(3)\sin(b\log 3) + f(4)\sin(b\log 4) - f(5)\sin(b\log 5) + \cdots\cdots \tag{10}$$

We can have the following (11) regarding real number x from the above (9) and (10) with the method shown in item 1.3 of [Appendix 1]. And the value of (11) is always zero at any value of x.

$$\begin{aligned} 0 \equiv& \cos x\{\text{right side of } (9)\} + \sin x\{\text{right side of } (10)\} \\ =& \cos x\{f(2)\cos(b\log 2) - f(3)\cos(b\log 3) + f(4)\cos(b\log 4) - \cdots\cdots\} \\ &+ \sin x\{f(2)\sin(b\log 2) - f(3)\sin(b\log 3) + f(4)\sin(b\log 4) - \cdots\cdots\} \\ =& f(2)\cos(b\log 2 - x) - f(3)\cos(b\log 3 - x) + f(4)\cos(b\log 4 - x) \\ &- f(5)\cos(b\log 5 - x) + f(6)\cos(b\log 6 - x) - \cdots\cdots \end{aligned} \tag{11}$$

At $a = 0$ we have the following (8-1) and the above (11) holds at $a = 0$.

$$f(n) = \frac{1}{n^{1/2-a}} - \frac{1}{n^{1/2+a}} \equiv 0 \qquad (n = 2, 3, 4, 5, \cdots\cdots) \tag{8-1}$$

We have the following (12-1) by substituting $b\log 1$ for x in (11).

$$\begin{aligned} 0 =& f(2)\cos(b\log 2 - b\log 1) - f(3)\cos(b\log 3 - b\log 1) + f(4)\cos(b\log 4 - b\log 1) \\ &- f(5)\cos(b\log 5 - b\log 1) + f(6)\cos(b\log 6 - b\log 1) - \cdots\cdots \end{aligned} \tag{12-1}$$

We have the following (12-2) by substituting $b\log 2$ for x in (11).

$$\begin{aligned} 0 =& f(2)\cos(b\log 2 - b\log 2) - f(3)\cos(b\log 3 - b\log 2) + f(4)\cos(b\log 4 - b\log 2) \\ &- f(5)\cos(b\log 5 - b\log 2) + f(6)\cos(b\log 6 - b\log 2) - \cdots\cdots \end{aligned} \tag{12-2}$$

We have the following (12-3) by substituting $b\log 3$ for x in (11).

$$\begin{aligned}0 =& f(2)\cos(b\log 2 - b\log 3) - f(3)\cos(b\log 3 - b\log 3) + f(4)\cos(b\log 4 - b\log 3)\\ &- f(5)\cos(b\log 5 - b\log 3) + f(6)\cos(b\log 6 - b\log 3) - \cdots\cdots \end{aligned} \tag{12-3}$$

In the same way as above we can have the following (12-N) by substituting $b\log N$ for x in (11). $(N = 4, 5, 6, 7, \cdots\cdots)$

$$\begin{aligned}0 =& f(2)\cos(b\log 2 - b\log N) - f(3)\cos(b\log 3 - b\log N) + f(4)\cos(b\log 4 - b\log N)\\ &- f(5)\cos(b\log 5 - b\log N) + f(6)\cos(b\log 6 - b\log N) - \cdots\cdots \end{aligned} \tag{12-N}$$

3. The solution for the identity of (11)

We define $g(k, N)$ as follows. $(k = 2, 3, 4, 5, \cdots\cdots \qquad N = 1, 2, 3, 4, \cdots\cdots)$

$$\begin{aligned}g(k,N) &= \cos(b\log k - b\log 1) + \cos(b\log k - b\log 2) + \cos(b\log k - b\log 3) + \cdots + \cos(b\log k - b\log N)\\ &= \cos(b\log 1 - b\log k) + \cos(b\log 2 - b\log k) + \cos(b\log 3 - b\log k) + \cdots + \cos(b\log N - b\log k)\\ &= \cos(b\log 1/k) + \cos(b\log 2/k) + \cos(b\log 3/k) + \cdots + \cos(b\log N/k) \end{aligned} \tag{13}$$

We can have the following (14) from the equations of (12-1), (12-2), (12-3), $\cdots\cdots$, (12-N) with the method shown in item 1.4 of [Appendix 1].

$$\begin{aligned}0 = & f(2)\{\cos(b\log 2 - b\log 1) + \cos(b\log 2 - b\log 2) + \cos(b\log 2 - b\log 3) + \cdots + \cos(b\log 2 - b\log N)\}\\ &- f(3)\{\cos(b\log 3 - b\log 1) + \cos(b\log 3 - b\log 2) + \cos(b\log 3 - b\log 3) + \cdots + \cos(b\log 3 - b\log N)\}\\ &+ f(4)\{\cos(b\log 4 - b\log 1) + \cos(b\log 4 - b\log 2) + \cos(b\log 4 - b\log 3) + \cdots + \cos(b\log 4 - b\log N)\}\\ &- f(5)\{\cos(b\log 5 - b\log 1) + \cos(b\log 5 - b\log 2) + \cos(b\log 5 - b\log 3) + \cdots + \cos(b\log 5 - b\log N)\}\\ &+ \cdots\cdots\\ &= f(2)g(2,N) - f(3)g(3,N) + f(4)g(4,N) - f(5)g(5,N) + \cdots\cdots \end{aligned} \tag{14}$$

If (11) holds, the sum of the right sides of infinite number equations of (12-1), (12-2), (12-3), (12-4), (12-5), $\cdots\cdots$ becomes zero. The rightmost side of (14) is the sum of the right sides of N equations of (12-1), (12-2), (12-3),$\cdots\cdots$, (12-N) as shown in item 1.4 of [Appendix 1]. Thererfore if (11) holds, $\lim_{N\to\infty}$ {the rightmost side of (14)} $= 0$ must hold. Here we define $F(a)$ as follows.

$$F(a) = f(2) - f(3) + f(4) - f(5) + \cdots\cdots \tag{15}$$

We have the following (25) in [Appendix 2 : Investigation of $g(k, N)$].

$$g(k,N) \quad \sim \quad \frac{N\cos(b\log N)}{\sqrt{1+b^2}} \qquad (N \to \infty \quad k = 2, 3, 4, 5, \cdots\cdots) \tag{25}$$

From the above (15) and (25) we have the following (16).

$$
\begin{aligned}
&\text{The rightmost side of (14)}\\
&= f(2)g(2,N) - f(3)g(3,N) + f(4)g(4,N) - f(5)g(5,N) + \cdots\cdots\\
&\sim \quad f(2)\frac{N\cos(b\log N)}{\sqrt{1+b^2}} - f(3)\frac{N\cos(b\log N)}{\sqrt{1+b^2}} + f(4)\frac{N\cos(b\log N)}{\sqrt{1+b^2}}\\
&\qquad - f(5)\frac{N\cos(b\log N)}{\sqrt{1+b^2}} + \cdots\cdots\\
&= \frac{N\cos(b\log N)}{\sqrt{1+b^2}}\{f(2) - f(3) + f(4) - f(5) + \cdots\cdots\}\\
&= F(a)\frac{N\cos(b\log N)}{\sqrt{1+b^2}} \qquad (N \to \infty) \qquad (16)
\end{aligned}
$$

We have the following (17) by summarizing the above (16).

$$\text{The rightmost side of (14)} \quad \sim \quad F(a)\frac{N\cos(b\log N)}{\sqrt{1+b^2}} \quad (N \to \infty) \qquad (17)$$

$\lim_{N\to\infty} \frac{N\cos(b\log N)}{\sqrt{1+b^2}}$ diverges to $\pm\infty$. $0 < F(a)$ holds in $0 < a < 1/2$ as shown in [Appendix 3 : Investigation of $F(a)$]. Then $\lim_{N\to\infty}$ {the rightmost side of (14)} diverges to $\pm\infty$ in $0 < a < 1/2$ from the above (17). This shows (11) does not hold in $0 < a < 1/2$. (11) holds at $a = 0$ as shown in item 2. Therefore non-trivial zero point of Riemann zeta function $\zeta(s)$ does not exist in $0 < a < 1/2$ but only at $a = 0$.

4. Conclusion

a has the range of $0 \le a < 1/2$ by the critical strip of $\zeta(s)$. However, a cannot have any value but zero as shown in the above item 3. Therefore non-trivial zero point of Riemann zeta function $\zeta(s)$ shown by (2) and (3) must be $1/2 \pm bi$.

Appendix 1. : Equation construction

We can construct (9), (10), (11) and (14) by applying the following Theorem 1[1].

Theorem 1

If the following (Series 1) and (Series 2) converge respectively, the following (Series 3) and (Series 4) hold.

$$\text{(Series 1)} = a_1 + a_2 + a_3 + a_4 + a_5 + \cdots\cdots = A$$
$$\text{(Series 2)} = b_1 + b_2 + b_3 + b_4 + b_5 + \cdots\cdots = B$$
$$\text{(Series 3)} = (a_1 + b_1) + (a_2 + b_2) + (a_3 + b_3) + (a_4 + b_4) + \cdots\cdots = A + B$$
$$\text{(Series 4)} = (a_1 - b_1) + (a_2 - b_2) + (a_3 - b_3) + (a_4 - b_4) + \cdots\cdots = A - B$$

1.1. Construction of (9)

We can have (9) as (Series 4) by regarding (6) and (4) as (Series 1) and (Series 2) respectively.

1.2. Construction of (10)

We can have (10) as (Series 4) by regarding (7) and (5) as (Series 1) and (Series 2) respectively.

1.3. Construction of (11)

We can have (11) as (Series 3) by regarding the following (11-1) and (11-2) as (Series 1) and (Series 2) respectively.

$$\text{(Series 1)} = \cos x\{\text{right side of } (9)\} \equiv 0 \tag{11-1}$$
$$\text{(Series 2)} = \sin x\{\text{right side of } (10)\} \equiv 0 \tag{11-2}$$

1.4. Construction of (14)

1.4.1 We can have the following (12-1*2) as (Series 3) by regarding the following (12-1) and (12-2) as (Series 1) and (Series 2) respectively.

$$\begin{aligned}\text{(Series 1)} =& f(2)\cos(b\log 2 - b\log 1) - f(3)\cos(b\log 3 - b\log 1)\\ &+ f(4)\cos(b\log 4 - b\log 1) - f(5)\cos(b\log 5 - b\log 1)\\ &+ f(6)\cos(b\log 6 - b\log 1) - \cdots\cdots = 0\end{aligned} \tag{12-1}$$

$$\begin{aligned}\text{(Series 2)} =& f(2)\cos(b\log 2 - b\log 2) - f(3)\cos(b\log 3 - b\log 2)\\ &+ f(4)\cos(b\log 4 - b\log 2) - f(5)\cos(b\log 5 - b\log 2)\\ &+ f(6)\cos(b\log 6 - b\log 2) - \cdots\cdots = 0\end{aligned} \tag{12-2}$$

$$\begin{aligned}\text{(Series 3)} =& f(2)\{\cos(b\log 2 - b\log 1) + \cos(b\log 2 - b\log 2)\}\\ &- f(3)\{\cos(b\log 3 - b\log 1) + \cos(b\log 3 - b\log 2)\}\\ &+ f(4)\{\cos(b\log 4 - b\log 1) + \cos(b\log 4 - b\log 2)\}\\ &- f(5)\{\cos(b\log 5 - b\log 1) + \cos(b\log 5 - b\log 2)\}\\ &+ \cdots\cdots = 0 + 0\end{aligned} \tag{12-1*2}$$

1.4.2 We can have the following (12-1*3) as (Series 3) by regarding the above (12-1*2) and the following (12-3) as (Series 1) and (Series 2) respectively.

$$\begin{aligned}\text{(Series 2)} = {} & f(2)\cos(b\log 2 - b\log 3) - f(3)\cos(b\log 3 - b\log 3)\\ & + f(4)\cos(b\log 4 - b\log 3) - f(5)\cos(b\log 5 - b\log 3)\\ & + f(6)\cos(b\log 6 - b\log 3) - \cdots\cdots = 0 \qquad \text{(12-3)}\end{aligned}$$

$$\begin{aligned}&\text{(Series 3)}\\ &= f(2)\{\cos(b\log 2 - b\log 1) + \cos(b\log 2 - b\log 2) + \cos(b\log 2 - b\log 3)\}\\ &\quad - f(3)\{\cos(b\log 3 - b\log 1) + \cos(b\log 3 - b\log 2) + \cos(b\log 3 - b\log 3)\}\\ &\quad + f(4)\{\cos(b\log 4 - b\log 1) + \cos(b\log 4 - b\log 2) + \cos(b\log 4 - b\log 3)\}\\ &\quad - f(5)\{\cos(b\log 5 - b\log 1) + \cos(b\log 5 - b\log 2) + \cos(b\log 5 - b\log 3)\}\\ &\quad + \cdots\cdots = 0 + 0 \qquad \text{(12-1*3)}\end{aligned}$$

1.4.3 We can have the following (12-1*4) as (Series 3) by regarding the above (12-1*3) and the following (12-4) as (Series 1) and (Series 2) respectively.

$$\begin{aligned}\text{(Series 2)} = {} & f(2)\cos(b\log 2 - b\log 4) - f(3)\cos(b\log 3 - b\log 4)\\ & + f(4)\cos(b\log 4 - b\log 4) - f(5)\cos(b\log 5 - b\log 4)\\ & + f(6)\cos(b\log 6 - b\log 4) - \cdots\cdots = 0 \qquad \text{(12-4)}\end{aligned}$$

$$\begin{aligned}&\text{(Series 3)}\\ &= f(2)\{\cos(b\log 2 - b\log 1) + \cos(b\log 2 - b\log 2) + \cos(b\log 2 - b\log 3) + \cos(b\log 2 - b\log 4)\}\\ &\quad - f(3)\{\cos(b\log 3 - b\log 1) + \cos(b\log 3 - b\log 2) + \cos(b\log 3 - b\log 3) + \cos(b\log 3 - b\log 4)\}\\ &\quad + f(4)\{\cos(b\log 4 - b\log 1) + \cos(b\log 4 - b\log 2) + \cos(b\log 4 - b\log 3) + \cos(b\log 4 - b\log 4)\}\\ &\quad - f(5)\{\cos(b\log 5 - b\log 1) + \cos(b\log 5 - b\log 2) + \cos(b\log 5 - b\log 3) + \cos(b\log 5 - b\log 4)\}\\ &\quad + \cdots\cdots = 0 + 0 \qquad \text{(12-1*4)}\end{aligned}$$

1.4.4 In the same way as above we can have the following (12-1*N)=(14) as (Series 3) by regarding (12-1*N-1) and (12-N) as (Series 1) and (Series 2) respectively. $(N = 5, 6, 7, 8, \cdots\cdots)$ $g(k, N)$ is defined in page 3. $(k = 2, 3, 4, 5, \cdots\cdots)$

$$\begin{aligned}&\text{(Series 3)}=\\ &f(2)\{\cos(b\log 2 - b\log 1) + \cos(b\log 2 - b\log 2) + \cos(b\log 2 - b\log 3) + \cdots + \cos(b\log 2 - b\log N)\}\\ &-f(3)\{\cos(b\log 3 - b\log 1) + \cos(b\log 3 - b\log 2) + \cos(b\log 3 - b\log 3) + \cdots + \cos(b\log 3 - b\log N)\}\\ &+f(4)\{\cos(b\log 4 - b\log 1) + \cos(b\log 4 - b\log 2) + \cos(b\log 4 - b\log 3) + \cdots + \cos(b\log 4 - b\log N)\}\\ &-f(5)\{\cos(b\log 5 - b\log 1) + \cos(b\log 5 - b\log 2) + \cos(b\log 5 - b\log 3) + \cdots + \cos(b\log 5 - b\log N)\}\\ &+\cdots\cdots\end{aligned}$$

$$\begin{aligned}&= f(2)g(2,N) - f(3)g(3,N) + f(4)g(4,N) - f(5)g(5,N) + \cdots\cdots\\ &= 0 + 0 \qquad \text{(12-1*N)}\end{aligned}$$

Appendix 2. : Investigation of $g(k, N)$

2.1 We define G and H as follows. $(N = 1, 2, 3, 4, \cdots\cdots)$

$$G = \lim_{N\to\infty} \frac{1}{N}\{\cos(b\log\frac{1}{N}) + \cos(b\log\frac{2}{N}) + \cos(b\log\frac{3}{N}) + \cdots + \cos(b\log\frac{N}{N})\}$$
$$= \int_0^1 \cos(b\log x)dx \tag{20-1}$$
$$H = \lim_{N\to\infty} \frac{1}{N}\{\sin(b\log\frac{1}{N}) + \sin(b\log\frac{2}{N}) + \sin(b\log\frac{3}{N}) + \cdots + \sin(b\log\frac{N}{N})\}$$
$$= \int_0^1 \sin(b\log x)dx \tag{20-2}$$

We calculate G and H by Integration by parts.

$$G = [x\cos(b\log x)]_0^1 + bH = 1 + bH$$
$$H = [x\sin(b\log x)]_0^1 - bG = -bG$$

Then we can have the values of G and H from the above equations as follows.

$$G = \frac{1}{1+b^2} \qquad H = \frac{-b}{1+b^2} \tag{21}$$

2.2 We define $E_c(N)$ and $E_s(N)$ as follows.

$$\frac{\cos(b\log\frac{1}{N}) + \cos(b\log\frac{2}{N}) + \cos(b\log\frac{3}{N}) + \cdots + \cos(b\log\frac{N}{N})}{N} - G = E_c(N) \tag{22-1}$$

$$\frac{\sin(b\log\frac{1}{N}) + \sin(b\log\frac{2}{N}) + \sin(b\log\frac{3}{N}) + \cdots + \sin(b\log\frac{N}{N})}{N} - H = E_s(N) \tag{22-2}$$

From (20-1), (20-2), (22-1) and (22-2) we have the following (23).

$$\lim_{N\to\infty} E_c(N) = 0 \qquad \lim_{N\to\infty} E_s(N) = 0 \tag{23}$$

2.3 From (13) we can calculate $g(k, N)$ as follows. $(N = 1, 2, 3, 4, \cdots\cdots)$

$$g(k,N) = \cos(b\log 1/k) + \cos(b\log 2/k) + \cos(b\log 3/k) + \cdots + \cos(b\log N/k)$$
$$= N\tfrac{1}{N}\{\cos(b\log\tfrac{1}{N}\tfrac{N}{k}) + \cos(b\log\tfrac{2}{N}\tfrac{N}{k}) + \cos(b\log\tfrac{3}{N}\tfrac{N}{k}) + \cdots + \cos(b\log\tfrac{N}{N}\tfrac{N}{k})\}$$
$$= N\tfrac{1}{N}\{\cos(b\log\tfrac{1}{N} + b\log\tfrac{N}{k}) + \cos(b\log\tfrac{2}{N} + b\log\tfrac{N}{k})$$
$$+ \cos(b\log\tfrac{3}{N} + b\log\tfrac{N}{k}) + \cdots\cdots + \cos(b\log\tfrac{N}{N} + b\log\tfrac{N}{k})\}$$
$$= N\tfrac{1}{N}\cos(b\log\tfrac{N}{k})\{\cos(b\log\tfrac{1}{N}) + \cos(b\log\tfrac{2}{N}) + \cos(b\log\tfrac{3}{N}) + \cdots + \cos(b\log\tfrac{N}{N})\}$$
$$-N\tfrac{1}{N}\sin(b\log\tfrac{N}{k})\{\sin(b\log\tfrac{1}{N}) + \sin(b\log\tfrac{2}{N}) + \sin(b\log\tfrac{3}{N}) + \cdots + \sin(b\log\tfrac{N}{N})\}$$
$$= N\cos(b\log\tfrac{N}{k})G$$

$$+N\cos(b\log\tfrac{N}{k})\{\tfrac{\cos(b\log 1/N)+\cos(b\log 2/N)+\cos(b\log 3/N)+\cdots+\cos(b\log N/N)}{N}-G\}$$
$$-N\sin(b\log\tfrac{N}{k})H$$
$$-N\sin(b\log\tfrac{N}{k})\{\tfrac{\sin(b\log 1/N)+\sin(b\log 2/N)+\sin(b\log 3/N)+\cdots+\sin(b\log N/N)}{N}-H\} \quad (24\text{-}1)$$

$$= N\cos(b\log\frac{N}{k})G + N\cos(b\log\frac{N}{k})E_c(N) - N\sin(b\log\frac{N}{k})H$$
$$- N\sin(b\log\frac{N}{k})E_s(N) \quad (24\text{-}2)$$
$$= N\cos(b\log\frac{N}{k})\frac{1}{1+b^2} + N\cos(b\log\frac{N}{k})E_c(N)$$
$$+ N\sin(b\log\frac{N}{k})\frac{b}{1+b^2} - N\sin(b\log\frac{N}{k})E_s(N) \quad (24\text{-}3)$$
$$= \frac{N}{\sqrt{1+b^2}}\{\cos(b\log\frac{N}{k})\frac{1}{\sqrt{1+b^2}} + \sin(b\log\frac{N}{k})\frac{b}{\sqrt{1+b^2}}\}$$
$$+ N\cos(b\log\frac{N}{k})E_c(N) - N\sin(b\log\frac{N}{k})E_s(N) \quad (24\text{-}4)$$
$$= N\{\frac{\cos(b\log N/k - \tan^{-1} b)}{\sqrt{1+b^2}}$$
$$+ \cos(b\log\frac{N}{k})E_c(N) - \sin(b\log\frac{N}{k})E_s(N)\} \quad (24\text{-}5)$$
$$= N[\frac{1}{\sqrt{1+b^2}}\cos\{b\log N(1-\frac{\log k}{\log N}-\frac{\tan^{-1} b}{b\log N})\}$$
$$+ \cos(b\log\frac{N}{k})E_c(N) - \sin(b\log\frac{N}{k})E_s(N)] \quad (24\text{-}6)$$

From (22-1), (22-2) and (24-1) we have (24-2). From (21) and (24-2) we have (24-3).

2.4 From (23) and the above (24-6) we have the following (25).

$$g(k,N) \quad \sim \quad \frac{N\cos(b\log N)}{\sqrt{1+b^2}} \quad (N\to\infty \quad k=2,3,4,5,\cdots\cdots) \quad (25)$$

Appendix 3. : Investigation of $F(a)$

3.1 $F(0) = 0$ holds due to $f(n) \equiv 0$ at $a = 0$. The alternating series $F(a)$ converges due to $\lim_{n\to\infty} f(n) = 0$.

$$f(n) = \frac{1}{n^{1/2-a}} - \frac{1}{n^{1/2+a}} \geq 0 \quad (n = 2, 3, 4, 5, \cdots\cdots \quad 0 \leq a < 1/2) \tag{8}$$

$$F(a) = f(2) - f(3) + f(4) - f(5) + f(6) - \cdots\cdots \tag{16}$$

We have the following (31) by differentiating $f(n)$ regarding n.

$$\frac{df(n)}{dn} = \frac{1/2+a}{n^{a+3/2}} - \frac{1/2-a}{n^{3/2-a}} = \frac{1/2+a}{n^{a+3/2}}\{1 - (\frac{1/2-a}{1/2+a})n^{2a}\} \tag{31}$$

The value of $f(n)$ increases with increase of n and reaches the maximum value $f(n_{max})$ at $n = n_{max}$. Afterward $f(n)$ decreases to zero with $n \to \infty$. n_{max} is one of the 2 consecutive natural numbers that sandwich $\left(\frac{1/2+a}{1/2-a}\right)^{\frac{1}{2a}}$. (Graph 1) shows $f(n)$ in various value of a.

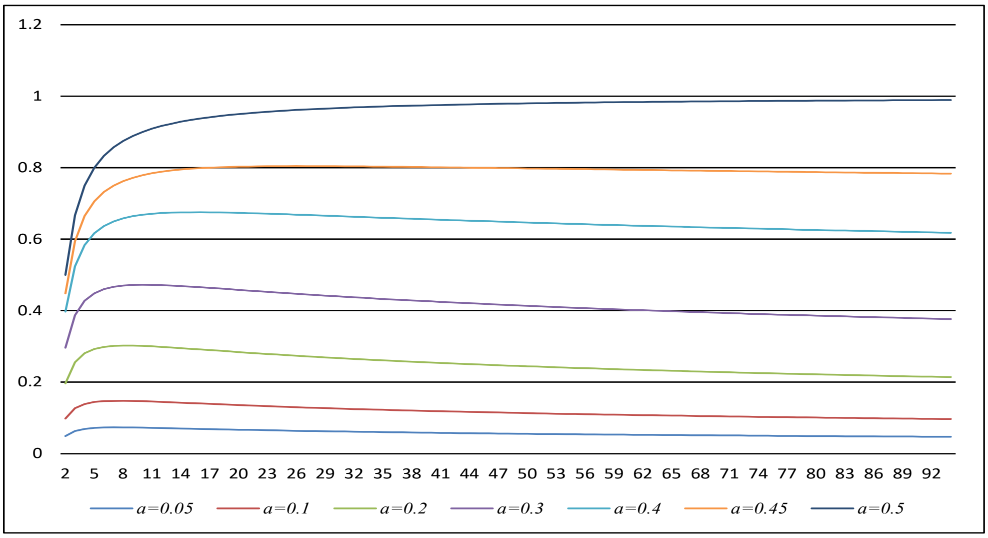

Figure 1. Graph 1 : $f(n)$ in various value of a

3.2 We define $F(a,n)$ as the following (32).

$$F(a,n) = f(2) - f(3) + f(4) - f(5) + \cdots + (-1)^n f(n) \tag{32}$$

$$\lim_{n\to\infty} F(a,n) = F(a) \tag{33}$$

$F(a)$ is an alternating series. So $F(a,n)$ repeats increase and decrease by $f(n)$ with increase of n as shown in (Graph 2). In (Graph 2) upper points mean $F(a, 2m)$ $\quad (m = 1, 2, 3, \cdots\cdots)$ and lower points mean $F(a, 2m+1)$. $F(a, 2m)$ decreases and converges to $F(a)$ with $m \to \infty$. $F(a, 2m+1)$ increases and also converges to $F(a)$ with $m \to \infty$ due to $\lim_{n\to\infty} f(n) = 0$. From the above (33) we have the following (34).

$$\lim_{m\to\infty} F(a, 2m) = \lim_{m\to\infty} F(a, 2m+1) = F(a) \tag{34}$$

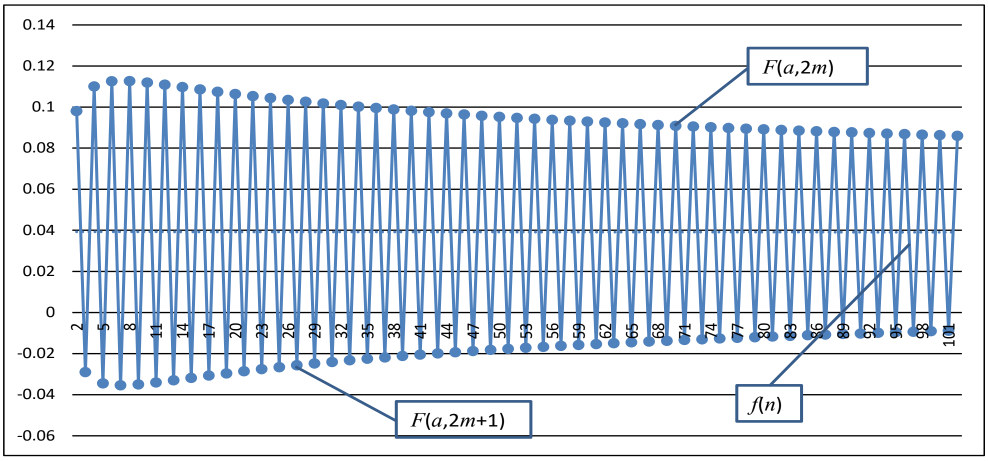

Figure 2. Graph 2 : $F(0.1, n)$ from 1st to 100th term

3.3 From the above (34) we can approximate $F(a)$ with the average of $\{F(a,n) + F(a, n+1)\}/2$. But we approximate $F(a)$ by the following (35) for better accuracy.

$$\frac{\frac{F(a,n-1)+F(a,n)}{2} + \frac{F(a,n)+F(a,n+1)}{2}}{2} = F(a)_n \tag{35}$$

We have the following (35-1) and (35-2) from the above (33) and (35).

$$\lim_{n\to\infty} F(a)_n = F(a) \tag{35-1}$$

$$F(a)_{n+1} = F(a)_n + (-1)^n \frac{\frac{f(n+2)-f(n+1)}{2} - \frac{f(n+1)-f(n)}{2}}{2} \tag{35-2}$$

3.3.1 (Graph 3) in the next page shows $F(a)_n$ calculated at 3 cases of $n = 500, 1000, 5000$. 3 line graphs overlap. Because the values of $F(a)_n$ calculated at 3 cases are equal to 4 digits after the decimal point. Therefore the values of (Table 1) are true as the values of $F(a)$ to 4 digits after the decimal point except $F(1/2)$.

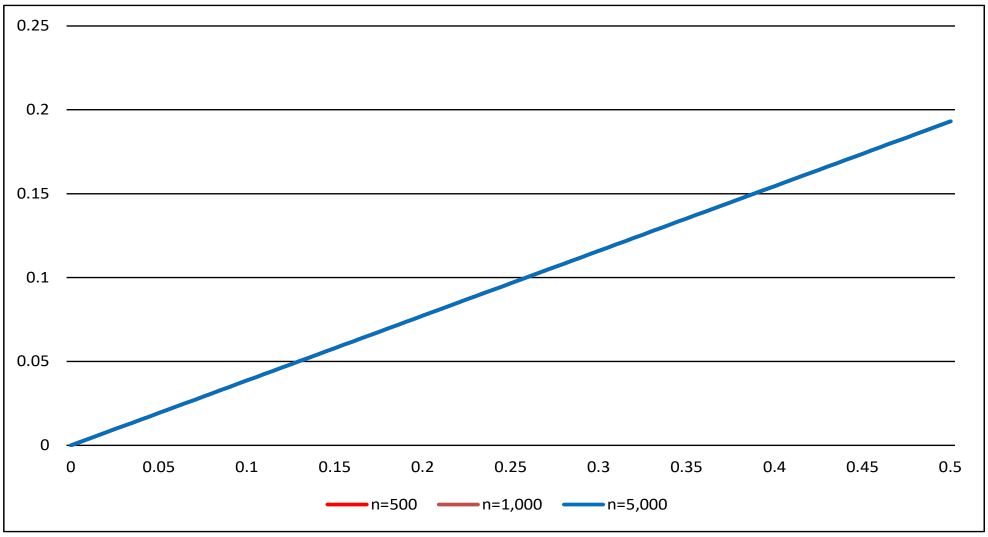

Figure 3. Graph 3 : $F(a)_n$ at 3 cases

a	0	0.05	0.1	0.15	0.2	0.25	0.3	0.35	0.4	0.45	0.5
n=500	0	0.01932876	0.03865677	0.05798326	0.0773074	0.09662832	0.11594507	0.13525658	0.15456168	0.17385904	0.19314718
n=1,000	0	0.01932681	0.03865282	0.05797725	0.0772993	0.09661821	0.11593325	0.13524382	0.15454955	0.17385049	0.19314743
n=5,000	0	0.01932876	0.03865676	0.05798324	0.07730738	0.09662829	0.11594504	0.13525655	0.15456165	0.17385902	0.19314718

Figure 4. Table 1 : The values of $F(a)_n$ at 3 cases

3.3.2 The range of a is $0 \leq a < 1/2$. $a = 1/2$ is not included in the range. But we added $F(1/2)_n$ to calculation due to the following reason. $f(n)$ at $a = 1/2$ is $1 - 1/n$ and $F(1/2)$ fluctuates due to $\lim_{n \to \infty} f(n) = 1$. The above (35-2) shows that $F(a)_n$ is partial sum of alternating series which has the term of $\dfrac{\frac{f(n+2)-f(n+1)}{2} - \frac{f(n+1)-f(n)}{2}}{2}$. Then $\lim_{n \to \infty} F(1/2)_n$ can converge to the fixed value on the condition of $\lim_{n \to \infty} \{f(n+1) - f(n)\} = 0$. The condition holds due to $f(n+1) - f(n) = 1/(n + n^2)$.

3.3.3 (Graph 3) is plotted by calculating $F(a)_n$ for a every 0.001. If $F(a)_n$ has a rapid convex (a combination of rapid decrease and rapid increase) or a rapid concave (a combination of rapid increase and rapid decrease) between $a = a_0$ and $a = a_0 + 0.001$ with increase of a, this rapid change is not displayed in (Graph 3). $(a_0 = 0, 0.001, 0.002, 0.003, \cdots, 0.497, 0.498, 0.499)$ But such a rapid change does not exist due to the following reason. Thererfore (Graph 3) shows $F(a)$ correctly except $F(1/2)$.

3.3.3.1 $f(n)$ has the following properties.

(1) $f(n) = 0$ holds at $a = 0$.

(2) $f(n)$ increases monotonically from 0 to $1 - 1/n$ with increase of a in

$0 \le a < 1/2$ from the following (36-1).

$$\frac{df(n)}{da} = f'(n) = \log n(\frac{1}{n^{1/2-a}} + \frac{1}{n^{1/2+a}}) > 0 \qquad (36\text{-}1)$$

(3) $f(n)$ is a strictly convex function regarding a in $0 < a < 1/2$ from the following (36-2).

$$\frac{d^2f(n)}{da^2} = f''(n) = (\log n)^2(\frac{1}{n^{1/2-a}} - \frac{1}{n^{1/2+a}}) \ge 0 \qquad (36\text{-}2)$$

3.3.3.2 We define n as even number because of $n = 500, 1000, 5000$. We also define $F(a,+)_n$ and $F(a,-)_n$ as follows.

$$F(a,+)_n = f(2) + f(4) + f(6) + \cdots + f(n-2) + (3/4)f(n) \qquad (37\text{-}1)$$
$$F(a,-)_n = f(3) + f(5) + f(7) + \cdots + f(n-1) + (1/4)f(n+1) \qquad (37\text{-}2)$$

We have the following (35-3) from (32), (35), (37-1) and (37-2).

$$\begin{aligned} F(a)_n =& f(2) - f(3) + f(4) - f(5) + \cdots + f(n-2) - f(n-1) \\ &+ (3/4)f(n) - (1/4)f(n+1) \\ =& F(a,+)_n - F(a,-)_n \end{aligned} \qquad (35\text{-}3)$$

$F(a,+)_n$ and $F(a,-)_n$ in the above (35-3) have the following properties respectively.

(1) $F(a,+)_n$ and $F(a,-)_n$ have the value of zero at $a = 0$ from the above item 3.3.3.1-(1) respectively.

(2) We have the follwing (38-1) and (38-2) from the above (37-1) and (37-2).

$$F'(a,+)_n = f'(2) + f'(4) + f'(6) + \cdots + f'(n-2) + (3/4)f'(n) \qquad (38\text{-}1)$$
$$F'(a,-)_n = f'(3) + f'(5) + f'(7) + \cdots + f'(n-1) + (1/4)f'(n+1) \qquad (38\text{-}2)$$

We have the follwing (38-3) from the above item 3.3.3.1-(2), (38-1) and (38-2).

$$F'(a,+)_n > 0 \qquad F'(a,-)_n > 0 \qquad (0 \le a < 1/2) \qquad (38\text{-}3)$$

$F(a,+)_n$ and $F(a,-)_n$ increase monotonically with increase of a in $0 \le a < 1/2$ from the above (38-3) respectively.

(3) We have the follwing (39-1) and (39-2) from the above (38-1) and (38-2).

$$F''(a,+)_n = f''(2) + f''(4) + f''(6) + \cdots + f''(n-2) + (3/4)f''(n) \qquad (39\text{-}1)$$

$$F''(a,-)_n = f''(3) + f''(5) + f''(7) + \cdots + f''(n-1) + (1/4)f''(n+1) \tag{39-2}$$

We have the follwing (39-3) from the above item 3.3.3.1-(3), (39-1) and (39-2).

$$F''(a,+)_n > 0 \qquad F''(a,-)_n > 0 \qquad (0 < a < 1/2) \tag{39-3}$$

$F(a,+)_n$ and $F(a,-)_n$ are strictly convex functions regarding a in $0 < a < 1/2$ from the above (39-3) respectively.

(4) $F(a,+)_n$ and $F(a,-)_n$ do not have a rapid convex or a rapid concave with increase of a between $a = a_0$ and $a = a_0 + 0.001$ from the above item (2) and (3) respectively.

3.3.3.3 $F(a)_n$ is the difference between $F(a,+)_n$ and $F(a,-)_n$ as shown in the above (35-3). Then $F(a)_n$ has the following properties.

(1) $F(a)_n = 0$ holds at $a = 0$ from the above item 3.3.3.2-(1).

(2) $F(a)_n$ does not have a rapid convex or a rapid concave with increase of a between $a = a_0$ and $a = a_0 + 0.001$ from the above item 3.3.3.2-(4).
($a_0 = 0, 0.001, 0.002, 0.003, \cdots, 0.497, 0.498, 0.499$)

3.4 We define as follows.

$$f'(n) = \frac{df(n)}{da} = \frac{1}{n^{1/2-a}}\log n + \frac{1}{n^{a+1/2}}\log n > 0 \tag{40}$$

$$F'(a) = f'(2) - f'(3) + f'(4) - f'(5) + \cdots\cdots \tag{41}$$

$$F'(a,n) = f'(2) - f'(3) + f'(4) - f'(5) + \cdots + (-1)^n f'(n) \tag{42}$$

$F'(a)$ is an alternating series. $F'(a)$ converges due to $\lim_{n\to\infty} f'(n) = 0$. We can calculate approximation of $F'(a)$ i.e. $F'(a)_n$ according to the following (43).

$$\frac{\frac{F'(a,n-1)+F'(a,n)}{2} + \frac{F'(a,n)+F'(a,n+1)}{2}}{2} = F'(a)_n \tag{43}$$

We have the following (43-1) and (43-2) from the above (42) and (43).

$$\lim_{n\to\infty} F'(a,n) = F'(a) \tag{43-1}$$

$$F'(a)_{n+1} = F'(a)_n + (-1)^n \frac{\frac{f'(n+2)-f'(n+1)}{2} - \frac{f'(n+1)-f'(n)}{2}}{2} \tag{43-2}$$

3.4.1 (Graph 4) shows $F'(a)_n$ calculated by the above (43) at 5 cases of n = 500, 1000, 2000, 5000, 10000. 5 line graphs overlap. Because the values of $F'(a)_n$ calculated at 5 cases are equal to 6 digits after the decimal point. Therefore the values of (Table 2) are true as the values of $F'(a)$ to 6 digits after the decimal point except $F'(1/2)$.

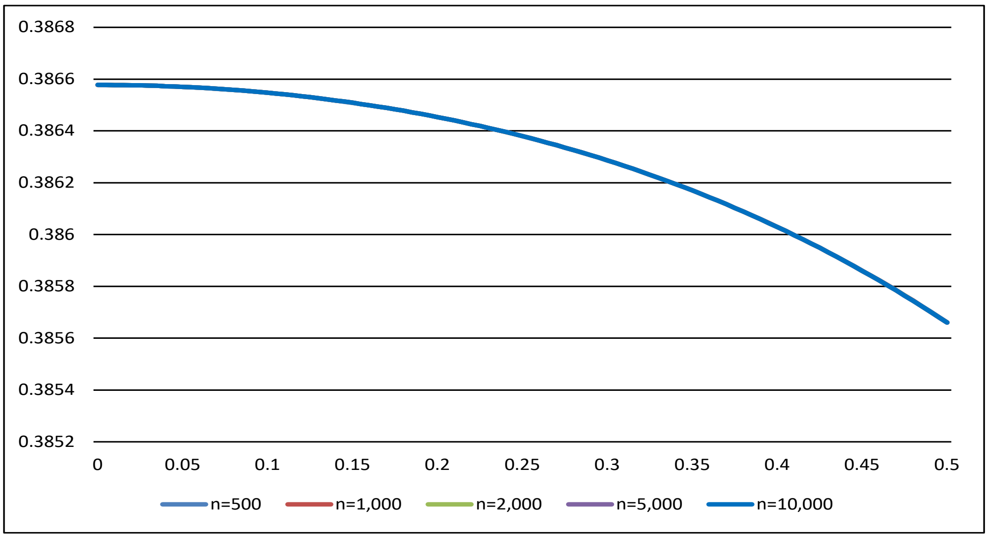

Figure 5. Graph 4 : $F'(a)_n$ at 5 cases

a	0	0.05	0.1	0.15	0.2	0.25	0.3	0.35	0.4	0.45	0.5
n=500	0.38657754	0.38657004	0.38654734	0.38650882	0.38645348	0.3863799	0.38628625	0.38617032	0.3860295	0.38586078	0.38566075
n=1,000	0.38657764	0.38657014	0.38654743	0.38650891	0.38645355	0.38637995	0.38628627	0.3861703	0.3860294	0.38586057	0.38566038
n=2,000	0.38657766	0.38657016	0.38654745	0.38650893	0.38645357	0.38637996	0.38628628	0.3861703	0.38602938	0.38586052	0.38566029
n=5,000	0.38657766	0.38657016	0.38654745	0.38650893	0.38645358	0.38637997	0.38628628	0.3861703	0.38602938	0.38586051	0.38566026
n=10,000	0.38657766	0.38657016	0.38654745	0.38650893	0.38645358	0.38637997	0.38628629	0.3861703	0.38602938	0.3858605	0.38566026

Figure 6. Table 2 : The values of $F'(a)_n$ at 5 cases

3.4.2 The range of a is $0 \leq a < 1/2$. $a = 1/2$ is not included in the range. But we added $F'(1/2)_n$ to calculation according to the following reason.
$f'(n)$ at $a = 1/2$ is $(1 + 1/n)\log n$ and $F'(1/2)$ diverges to $\pm\infty$ because $\lim_{n\to\infty}\{(1 + 1/n)\log n\}$ diverges to ∞. The above (43-2) shows that $F'(a)_n$ is partial sum of alternating series which has the term of $\dfrac{\frac{f'(n+2)-f'(n+1)}{2} - \frac{f'(n+1)-f'(n)}{2}}{2}$ and $\lim_{n\to\infty} F'(1/2)_n$ can converge to the fixed value on the condition of $\lim_{n\to\infty}\{f'(n+1)-f'(n)\} = 0$. $\lim_{n\to\infty}\{f'(n+1)-f'(n)\} = 0$ holds as follows.
$f'(n)$ at $a = 1/2$ is a monotonically increasing function regarding n due to $\dfrac{df'(n)}{dn} = \dfrac{1+n-\log n}{n^2} > 0$. Therefore $0 < f'(n+1) - f'(n)$ holds.

$$\begin{aligned} 0 &< f'(n+1) - f'(n) = \{1 + 1/(n+1)\}\log(n+1) - (1 + 1/n)\log n \\ &< (1 + 1/n)\log(n+1) - (1 + 1/n)\log n = (1 + 1/n)\log(1 + 1/n) \end{aligned}$$

From the above inequality we can have $\lim_{n\to\infty}\{f'(n+1) - f'(n)\} = 0$ due to $\lim_{n\to\infty}\{(1 + 1/n)\log(1 + 1/n)\} = 0$.

3.4.3 (Graph 4) is plotted by calculating $F'(a)_n$ for a every 0.001. If $F'(a)_n$ has a rapid convex (a combination of rapid decrease and rapid increase) or a rapid concave (a combination of rapid increase and rapid decrease) between $a = a_0$ and $a = a_0 + 0.001$ with increase of a, this rapid change is not displayed in (Graph 4). ($a_0 = 0, 0.001, 0.002, 0.003, \cdots, 0.497, 0.498, 0.499$) But such a rapid change does not exist due to the following reason. Thererfore (Graph 4) shows $F'(a)$ correctly except $F'(1/2)$.

3.4.3.1 $f'(n)$ has the following properties.

(1) $f'(n) = \dfrac{2}{\sqrt{n}} \log n$ holds at $a = 0$.

(2) $f'(n)$ increases monotonically from $\dfrac{2}{\sqrt{n}} \log n$ to $(1 + 1/n) \log n$ with increase of a in $0 < a < 1/2$ from the following (44-1).

$$\frac{df'(n)}{da} = (\log n)^2 \left(\frac{1}{n^{1/2-a}} - \frac{1}{n^{1/2+a}}\right) \geq 0 \tag{44-1}$$

(3) $f'(n)$ is a strictly convex function regarding a in $0 \leq a < 1/2$ from the following (44-2).

$$\frac{d^2 f'(n)}{da^2} = (\log n)^3 \left(\frac{1}{n^{1/2-a}} + \frac{1}{n^{1/2+a}}\right) > 0 \tag{44-2}$$

3.4.3.2 We can confirm that $F'(a)_n$ does not have a rapid convex or a rapid concave with increase of a between $a = a_0$ and $a = a_0 + 0.001$ through the same discussion as in item 3.3.3.
($a_0 = 0, 0.001, 0.002, 0.003, \cdots, 0.497, 0.498, 0.499$)

3.5 $0 < F(a)$ holds in $0 < a < 1/2$ due to the following reason.

3.5.1 $F(0) = 0$ holds as shown in item 3.1.

3.5.2 $F(a)$ is a monotonically increasing function in $0 \leq a < 1/2$ because $0 < F'(a)$ holds in $0 \leq a < 1/2$ as shown in (Graph 4).

References

[1] Yukio Kusunoki, Introduction to infinite series, Asakura syoten, (1972), page 22, (written in Japanese)

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