Sequences: Adventure of a Lifetime

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Abstract. Some formulas related to Pi

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Introduction

Recall that

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \ldots \tag{1}$$

An Integer Sequence

Define s_n , $n = 0, 1, 2, 3, \ldots$ by

$$s_0 = 1 \tag{2}$$

$$s_n = 2 \sum_{k=0}^{n-1} (-1)^k \binom{2n}{2k} \cdot (2n - 2k - 1) \cdot (2n - 2k) s_{n-k-1} \ , \ n = 1, 2, 3, \ldots \tag{3}$$

we have $s_n \in \mathbb{N} \ \forall \ n \in \mathbb{N} \cup \{0\}$.

$$s_n = \{1, 4, 72, 2940, 214\,928, 24\,756\,660, 4\,141\,462\,776, 950\,488\,227\,052, \ \ldots\} \tag{4}$$

remark: $\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$, $0 \le k \le n$.

Formulas

Entry 1.

$$\lambda = \lim_{n \longrightarrow \infty} \sqrt{\frac{(2n+1) \cdot (2n+2) \cdot s_n}{s_{n+1}}} = 0.892874 \ldots \tag{5}$$

Entry 2. for $i = \sqrt{-1}$ we have

$$H(x) = \frac{1 - x^3 \sin x}{2x^2 \cos x - 1} \tag{6}$$

$$\lambda = \frac{1}{2\pi} \int_0^{2\pi} e^{ix} H\left(\frac{1}{2} + \frac{1}{2} e^{ix}\right) dx \tag{7}$$

Entry 3. for $i = \sqrt{-1}$ we have

$$\lambda = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} e^{i x} H\left(e^{i x}\right) dx + \frac{2}{\pi} \int_0^1 \frac{1 - x^3 \sinh x}{2 x^2 \cosh x - 1} dx \tag{8}$$

$H(x)$ is defined by (6).

Entry 4.

$$\frac{1}{\lambda} = \sqrt{2 \cos\left(\frac{1}{\sqrt{2 \cos\left(\frac{1}{\sqrt{2 \ldots}}\right)}}\right)} \tag{9}$$

$$\lambda = \sqrt{\frac{1}{2} \sec \sqrt{\frac{1}{2} \sec \sqrt{\frac{1}{2} \ldots}}} \tag{10}$$

Entry 5.

$$\pi = 4 \sum_{n=0}^{\infty} \frac{2^{-2 n}}{2 n + 1} \binom{2 n}{n} \lambda^{2 n+1} \cos\left(\frac{(2 n + 1) \lambda}{2}\right) \tag{11}$$

Entry 6.

$$\pi = 2 \lambda + \ln\left(\frac{2 \lambda^2 + 1}{2 \lambda^2 - 1}\right) - \sum_{n=0}^{\infty} \frac{(\cos \lambda)^{2 n+3}}{2 n + 3} \sum_{k=0}^{n} \frac{2^{-2 k}}{k + 1} \binom{2 k}{k} \tag{12}$$

Entry 7.

$$\pi + 4 \sum_{n=0}^{\infty} \left(\frac{\lambda^4}{2}\right)^{n+1} \sum_{k=0}^{[n/2]} \frac{(-1)^k}{2 k + 1} \binom{n}{n - 2 k} = 8 \sum_{n=0}^{\infty} \frac{(-1)^n}{2 n + 1} \lambda^{4 n+2} \cos((2 n + 1) \lambda) \tag{13}$$

Entry 8.

$$\pi = 6 \lambda^2 \sum_{n=0}^{\infty} \sum_{k=[n/2]}^{n} \frac{(\lambda/2)^{4 k}}{2 k + 1} \binom{2 k}{k} \binom{2 k + 1}{2 k - n} \cos\left((2 n - 2 k + 1) \lambda\right) \tag{14}$$

Entry 9.

$$\pi = 2 \tan^{-1}\left(\frac{\tan \lambda}{2}\right) + 4 \sum_{n=0}^{\infty} \frac{(\cos \lambda)^{2 n+1}}{2 n + 1} \sin((2 n + 1) \lambda) \tag{15}$$

Entry 10.

$$\pi = 2 \tan^{-1}\left(\frac{\tan^2 \lambda}{2}\right) + 4 \sum_{n=0}^{\infty} \frac{(-1)^n (\cos \lambda)^{2 n+1}}{2 n + 1} \cos((2 n + 1) \lambda) \tag{16}$$

Entry 11.

$$\pi = 2 \lambda + 2 \sin \lambda \sum_{n=0}^{\infty} \left(\frac{1}{2 \lambda^2}\right)^{2 n+1} \sum_{k=0}^{n} \frac{(-1)^k}{2 k + 1} \binom{n}{n - k} \tag{17}$$

Entry 12.

$$\pi = 4\lambda - 4\sum_{n=0}^{\infty}(-1)^n\left(\frac{\tan\lambda - 1}{2}\right)^{n+1}\sum_{k=0}^{[n/2]}\frac{(-1)^k}{2k+1}\binom{n}{n-2k} \tag{18}$$

Entry 13.

$$x_0 = 0 \;\; , n = 0, 1, 2, 3, \ldots ; \; x_{n+1} = \sqrt{\frac{1}{2}\sec x_n} \implies x_n \longrightarrow \lambda \tag{19}$$

Entry 14.

$$2\lambda^2\cos\lambda - 1 = 0 \tag{20}$$

Entry 15. Zeros of $f(x) = 2x^2\cos x - 1 = 0$.

The equation $f(x) = 0$, $x > 0$, has an infinite number of solutions in x . the nth positive zeros of $f(x) = 0$ are denoted by λ_n , $n = 0, 1, 2, 3, \ldots.$

$$\lambda = \lambda_0 = 0.8928\ldots , \; \lambda_1 = 1.2390\ldots , \; \lambda_2 = 4.7346\ldots , \; \lambda_3 = 7.8458\ldots \tag{21}$$

Entry 16.

$$\frac{\pi}{2} = \lambda_0 + \sin^{-1}\left(\frac{1}{2\lambda_0^2}\right) \tag{22}$$

$$\frac{\pi}{2} = \lambda_1 + \sin^{-1}\left(\frac{1}{2\lambda_1^2}\right) \tag{23}$$

$$\frac{3\pi}{2} = \lambda_2 - \sin^{-1}\left(\frac{1}{2\lambda_2^2}\right) \tag{24}$$

$$\frac{5\pi}{2} = \lambda_3 + \sin^{-1}\left(\frac{1}{2\lambda_3^2}\right) \tag{25}$$

$$\frac{(2n-1)\pi}{2} = \lambda_n + (-1)^{n-1}\sin^{-1}\left(\frac{1}{2\lambda_n^2}\right) \;\; , n = 1, 2, 3, \ldots \tag{26}$$

Entry 17. for $n = 1, 2, 3, \ldots$ we have

$$y_{n,0} = 0 \;\; , \; y_{n,k+1} = \sin^{-1}\left(\frac{2}{((2n-1)\pi + 2(-1)^n y_{n,k})^2}\right) \;\; , \; k = 0, 1, 2, 3, \ldots \implies \lim_{k\longrightarrow\infty} y_{n,k} = \sin^{-1}\left(\frac{1}{2\lambda_n^2}\right) \tag{27}$$

Entry 18. for $n = 1, 2, 3, \ldots$ we have

$$\lambda_{n,0} = \frac{(2n-1)\pi}{2} \;\; , \; \lambda_{n,k+1} = \frac{(2n-1)\pi}{2} + (-1)^n\sin^{-1}\left(\frac{1}{2\lambda_{n,k}^2}\right) , \; k = 0, 1, 2, 3, \ldots \implies \lim_{k\longrightarrow\infty} \lambda_{n,k} = \lambda_n \tag{28}$$

Entry 19.

Define t_n , $n = 0, 1, 2, 3, \ldots$, by

$$t_0 = 1 , \; t_1 = \frac{3}{2} , \; t_n = \frac{3}{2}t_{n-1} - \sum_{k=2}^{n}\frac{E_k}{(2k)!}t_{n-k} \;\; , \; n = 2, 3, 4, \ldots \tag{29}$$

where E_k are the Euler numbers.

we have

$$t_n = \left\{1, \frac{3}{2}, \frac{49}{24}, \frac{1919}{720}, \frac{137537}{40320}, \frac{15687359}{36288001}, \frac{2611231393}{479001600}, \ldots\right\} \tag{30}$$

$$\lambda = \lim_{n\longrightarrow\infty} \sqrt{\frac{t_n}{t_{n+1}}} = 0.892874\ldots \tag{31}$$

Remark: $E_k = \{1, 5, 61, 1385, 50521, \ldots\}$.

Entry 20.

$$x_0 = 1\ , n = 0, 1, 2, 3, \ldots;\ x_{n+1} = \frac{1}{2\,x_n \cos x_n} \Longrightarrow x_n \longrightarrow \lambda \tag{32}$$

Entry 21.

$$\pi = 2\,\lambda + \frac{4}{3} \sum_{n=0}^{\infty} \left(\frac{3}{4\,\lambda^2}\right)^{2\,n+1} \sum_{k=0}^{\left[\frac{2\,n+1}{3}\right]} \binom{2\,n-2\,k}{n-k} \binom{2\,n-2\,k+1}{k} \frac{(-1)^k\,(16/27)^k}{2\,n-2\,k+1} \tag{33}$$

Entry 22.

$$x_0 = 1\ , n = 0, 1, 2, 3, \ldots;\ x_{n+1} = \cos\left(\frac{1}{\sqrt{2\,x_n}}\right) \Longrightarrow x_n \longrightarrow \cos\lambda \tag{34}$$

Entry 23.

Define r_n, $n = 0, 1, 2, 3, \ldots$, by

$$r_0 = 1\ ,\ r_n = \sum_{k=1}^{n} \frac{(-1)^{k-1}\,r_{n-k}}{2^{k-1}\,(2\,k-2)!}\ ,\ n = 1, 2, 3, \ldots \tag{35}$$

we have

$$r_n = \left\{1, 1, \frac{3}{4}, \frac{49}{96}, \frac{1919}{5760}, \frac{137537}{645120}, \ldots\right\} \tag{36}$$

$$\lambda = \lim_{n\longrightarrow\infty} \sqrt{\frac{r_n}{2\,r_{n+1}}} = 0.892874\ldots \tag{37}$$

Entry 24.

$$1 = 2 \sum_{n=1}^{\infty} (-1)^{n-1} J_{2\,n-1}(\lambda^2\,\pi) \cos((2\,n-1)\,\lambda) \tag{38}$$

$$0 = J_0(\lambda^2\,\pi) + 2 \sum_{n=1}^{\infty} (-1)^n J_{2\,n}(\lambda^2\,\pi) \cos(2\,n\,\lambda) \tag{39}$$

Remark: $J_n(x)$ is the Bessel function of the first kind.

Entry 25.

$$\lambda^2 = \sum_{n=1}^{\infty} \frac{2^{-n}}{n!}\,a_n \tag{40}$$

$$a_n = \lim_{x\longrightarrow 0} \frac{d^{n-1}}{dx^{n-1}} \left(\frac{1}{\cos\sqrt{x}}\right)^n \tag{41}$$

$$a_n = \left\{1, 1, \frac{11}{4}, \frac{188}{15}, \frac{26\,837}{336}, \frac{206\,056}{315}, \frac{622\,187\,543}{95\,040}, \ldots\right\} \tag{42}$$

Entry 26.

$$\frac{\pi}{4}\sin^{-1}\left(\frac{1}{2\,\lambda^2}\right) = \sum_{n=0}^{\infty} \frac{\cos((2\,n+1)\,\lambda)}{(2\,n+1)^2} \tag{43}$$

Entry 27.

$$x_0 = 0.8\,,\ n = 0, 1, 2, 3, \ldots,;\ x_{n+1} = \frac{2\,(x_n \cos x_n)^2 + \cos x_n - x_n \sin x_n}{4\,x_n (\cos x_n)^2 - \sin x_n} \Longrightarrow x_n \longrightarrow \lambda \tag{44}$$

$$|\lambda - x_n| \le 0.8\,|\lambda - x_{n-1}|^2\,,\ n = 1, 2, 3, \ldots \tag{45}$$

$$|\lambda - x_0| = 9.2\cdot 10^{-2}\,,\ |\lambda - x_1| = 2.6\cdot 10^{-3}\,,\ |\lambda - x_2| = 5.5\cdot 10^{-6} \tag{46}$$

$$|\lambda - x_3| = 2.4\cdot 10^{-11}\,,\ |\lambda - x_4| = 4.6\cdot 10^{-22}\,,\ |\lambda - x_5| = 1.6\cdot 10^{-43}\,,\ |\lambda - x_6| = 2.2\cdot 10^{-86} \tag{47}$$

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