The Proofs of Legendre's Conjecture and Three Related Conjectures

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Abstracts

In this paper, we are going to prove Legendre's Conjecture: There is a prime number between n^2 and $(n+1)^2$ for every positive integer n. We will also prove three related conjectures. The method that we use is to analyze a binomial coefficient. It has been developed from the method of analyzing a central binomial coefficient that was used by Paul Erdős to prove Bertrand's postulate - Chebyshev's theorem.

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1. Introduction

Legendre's Conjecture was proposed by Andrien-Marie Legendre [1]. The conjecture is one of Legendre's problems (1912) on prime numbers. It states that there is a prime number between n^2 and $(n+1)^2$ for every positive integer n.

In this paper, we will prove Legendre's Conjecture by analyzing the binomial coefficient $\binom{\lambda n}{n}$ where $\lambda \geq 3$ is an integer. It is developed from the method that was used by Paul Erdős [2] to prove Bertrand's postulate - Chebyshev's theorem [3].

In Section 1, we will define the prime number factorization operator and clarify some terms and concepts. In Section 2, we will derive some lemmas. In Section 3, we will develop a theorem to be used in the proofs of the conjectures in the later sections. In Section 4, we will prove Legendre's conjecture, and in Section 5, we will prove Oppermann's conjecture [4], Brocard's conjecture [5], and Andrica's conjecture [6].

Definition: $\Gamma_{a\geq p>b}\{\binom{\lambda n}{n}\}$ denotes the prime factorization operator of $\binom{\lambda n}{n}$. It is the product of the prime numbers in the decomposition of $\binom{\lambda n}{n}$ in the range of $a \geq p > b$. In this operator, p is a prime number, a and b are real numbers, and $\lambda n \geq a \geq p > b \geq 1$.

It has some properties:

It is always true that $\Gamma_{a\geq p>b}\{\binom{\lambda n}{n}\} \geq 1$ **— (1.1)**

If there is no prime number in $\Gamma_{a\geq p>b}\{\binom{\lambda n}{n}\}$, then $\Gamma_{a\geq p>b}\{\binom{\lambda n}{n}\} = 1$, or vice versa, if $\Gamma_{a\geq p>b}\{\binom{\lambda n}{n}\} = 1$, then there is no prime number in $\Gamma_{a\geq p>b}\{\binom{\lambda n}{n}\}$. **— (1.2)**

For example, when $\lambda = 5$ and $n = 4$, $\Gamma_{16\geq p>10}\{\binom{20}{4}\} = 13^0 \cdot 11^0 = 1$. No prime number 13 or 11 is in $\binom{20}{4}$ in the range of $16 \geq p > 10$.

If there is at least one prime number in $\Gamma_{a\geq p>b}\{\binom{\lambda n}{n}\}$, then $\Gamma_{a\geq p>b}\{\binom{\lambda n}{n}\} > 1$, or vice versa, if $\Gamma_{a\geq p>b}\{\binom{\lambda n}{n}\} > 1$, then there is at least one prime number in $\Gamma_{a\geq p>b}\{\binom{\lambda n}{n}\}$. **— (1.3)**

For example, when $\lambda = 5$ and $n = 4$, $\Gamma_{20\geq p>16}\{\binom{20}{4}\} = 19 \cdot 17 > 1$. Prime numbers 19 and 17 are in $\binom{20}{4}$ in the range of $20 \geq p > 16$.

Let $v_p(n)$ be the $p\text{-}adic\ valuation$ of n, the exponent of the highest power of p that divides n. Similar to Paul Erdős' paper [2], we define *R(p)* by the inequalities $p^{R(p)} \leq \lambda n < p^{R(p)+1}$, and determine the $p\text{-}adic\ valuation$ of $\binom{\lambda n}{n}$.

$v_p\left(\binom{\lambda n}{n}\right) = v_p((\lambda n)!) - v_p(((\lambda-1)n)!) - v_p(n!) = \sum_{i=1}^{R(p)}\left(\left\lfloor\frac{\lambda n}{p^i}\right\rfloor - \left\lfloor\frac{(\lambda-1)n}{p^i}\right\rfloor - \left\lfloor\frac{n}{p^i}\right\rfloor\right) \leq R(p)$

because for any real numbers a and b, the expression of $\lfloor a+b\rfloor - \lfloor a\rfloor - \lfloor b\rfloor$ is 0 or 1.

Thus, if p divides $\binom{\lambda n}{n}$, then $v_p\left(\binom{\lambda n}{n}\right) \leq R(p) \leq \log_p(\lambda n)$, or $p^{v_p\left(\binom{\lambda n}{n}\right)} \leq p^{R(p)} \leq \lambda n$ **— (1.4)**

And if $\lambda n \geq p > \lfloor\sqrt{\lambda n}\rfloor$, then $0 \leq v_p\left(\binom{\lambda n}{n}\right) \leq R(p) \leq 1$ **— (1.5)**

Let $\pi(n)$ be the number of distinct prime numbers less than or equal to n. Among the first six consecutive natural numbers are three prime numbers 2, 3 and 5. Then, for each additional six consecutive natural numbers, at most one can add two prime numbers, $p \equiv 1$ (MOD 6) and $p \equiv 5$ (MOD 6). Thus, $\pi(n) \le \left\lfloor\frac{n}{3}\right\rfloor+2 \le \frac{n}{3}+2$. Since some of $n \equiv 1$ (MOD 6) and $n \equiv 5$ (MOD 6) are not prime numbers, as the number counts increase, $\pi(n)$ reduces from $\left\lfloor\frac{n}{3}\right\rfloor+2$.

For $n \ge 24$, $\pi(n) \le \left\lfloor\frac{n}{3}\right\rfloor+1 \le \frac{n}{3}+1$ **— (1.6)**

From the prime number decomposition,

when $n > \lfloor\sqrt{\lambda n}\rfloor$, $\binom{\lambda n}{n} = \Gamma_{\lambda n \ge p > n}\{\frac{(\lambda n)!}{n!\cdot((\lambda-1)n)!}\} \cdot \Gamma_{n \ge p > \lfloor\sqrt{\lambda n}\rfloor}\{\frac{(\lambda n)!}{n!\cdot((\lambda-1)n)!}\} \cdot \Gamma_{\lfloor\sqrt{\lambda n}\rfloor \ge p}\{\frac{(\lambda n)!}{n!\cdot((\lambda-1)n)!}\}$

when $n \le \lfloor\sqrt{\lambda n}\rfloor$, $\binom{\lambda n}{n} \le \Gamma_{\lambda n \ge p > n}\{\frac{(\lambda n)!}{n!\cdot((\lambda-1)n)!}\} \cdot \Gamma_{\lfloor\sqrt{\lambda n}\rfloor \ge p}\{\frac{(\lambda n)!}{n!\cdot((\lambda-1)n)!}\}$

Thus, $\binom{\lambda n}{n} \le \Gamma_{\lambda n \ge p > n}\{\frac{(\lambda n)!}{n!\cdot((\lambda-1)n)!}\} \cdot \Gamma_{n \ge p > \lfloor\sqrt{\lambda n}\rfloor}\{\frac{(\lambda n)!}{n!\cdot((\lambda-1)n)!}\} \cdot \Gamma_{\lfloor\sqrt{\lambda n}\rfloor \ge p}\{\frac{(\lambda n)!}{n!\cdot((\lambda-1)n)!}\}$

$\Gamma_{\lambda n \ge p > n}\{\frac{(\lambda n)!}{n!\cdot((\lambda-1)n)!}\} = \Gamma_{\lambda n \ge p > n}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\}$ since all prime numbers in $n!$ do not appear in the range of $\lambda n \ge p > n$.

Referring to **(1.5)**, $\Gamma_{n \ge p > \lfloor\sqrt{\lambda n}\rfloor}\{\frac{(\lambda n)!}{n!\cdot((\lambda-1)n)!}\} \le \prod_{n \ge p} p$. It has been proved [7] that for $n \ge 3$, $\prod_{n \ge p} p < 2^{2n-3}$. Thus, for $n \ge 3$, $\Gamma_{n \ge p > \lfloor\sqrt{\lambda n}\rfloor}\{\frac{(\lambda n)!}{n!\cdot((\lambda-1)n)!}\} \le \prod_{n \ge p} p < 2^{2n-3}$.

Referred to **(1.4)** and **(1.6)**, $\Gamma_{\lfloor\sqrt{\lambda n}\rfloor \ge p}\{\frac{(\lambda n)!}{n!\cdot((\lambda-1)n)!}\} \le (\lambda n)^{\frac{\sqrt{\lambda n}}{3}+1}$ when $\lfloor\sqrt{\lambda n}\rfloor \ge 24$.

Thus, for $n \ge 3$ and $\lfloor\sqrt{\lambda n}\rfloor \ge 24$, $\binom{\lambda n}{n} < \Gamma_{\lambda n \ge p > n}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\} \cdot 2^{2n-3} \cdot (\lambda n)^{\frac{\sqrt{\lambda n}}{3}+1}$ **— (1.7)**

2. Lemmas

Lemma 1: If a real number $x \ge 3$, then $\frac{2(2x-1)}{x-1} > \left(\frac{x}{x-1}\right)^x$ **— (2.1)**

Proof:

Let $f_1(x) = \frac{2(2x-1)}{x-1}$, then $f_1{}'(x) = \frac{2(x-1)(2x-1)'-2(2x-1)(x-1)'}{(x-1)^2} = \frac{-2}{(x-1)^2} < 0$.

Thus $f_1(x)$ is a strictly decreasing function for $x > 1$.

Since $f_1(3) = 5$ and $\lim_{x\to\infty} f_1(x) = 4$, for $x \ge 3$, we have $5 \ge f_1(x) = \frac{2(2x-1)}{x-1} \ge 4$. **— (2.1.1)**

Let $f_2(x) = \left(\frac{x}{x-1}\right)^x$, then $f_2{}'(x) = \left(\left(\frac{x}{x-1}\right)^x\right)' = \left(e^{x\cdot\ln\frac{x}{x-1}}\right)' = e^{x\cdot\ln\frac{x}{x-1}} \cdot \left(x \cdot \ln\frac{x}{x-1}\right)'$

$f_2{}'(x) = \left(\frac{x}{x-1}\right)^x \cdot \left(\ln\frac{x}{x-1} + x \cdot \left(\ln\frac{x}{x-1}\right)'\right) = \left(\frac{x}{x-1}\right)^x \cdot \left(\ln\frac{x}{x-1} + x \cdot \frac{x-1}{x} \cdot \frac{x-1-x}{(x-1)^2}\right)$

$f_2{}'(x) = \left(\frac{x}{x-1}\right)^x \cdot \left(\ln\frac{x}{x-1} - \frac{1}{x-1}\right)$ **— (2.1.2)**

In **(2.1.2)**, $\frac{1}{x-1} = \frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3} + \frac{1}{x^4} + \frac{1}{x^5} + \frac{1}{x^6} + \cdots$ **— (2.1.3)**

Using the formula: $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \cdots$, we have

$\ln\frac{x}{x-1} = \ln\frac{1}{1+\frac{-1}{x}} = -\ln\left(1+\frac{-1}{x}\right) = \frac{1}{x} + \frac{1}{2x^2} + \frac{1}{3x^3} + \frac{1}{4x^4} + \frac{1}{5x^5} + \frac{1}{6x^6} + \cdots$ **— (2.1.4)**

Thus for $x \geq 3$, $\ln\frac{x}{x-1} - \frac{1}{x-1} < 0$ **— (2.1.5)**

Since $\left(\frac{x}{x-1}\right)^x$ is a positive number for $x \geq 3$, $f_2{}'(x) = \left(\frac{x}{x-1}\right)^x \cdot \left(\ln\frac{x}{x-1} - \frac{1}{x-1}\right) < 0$. **— (2.1.6)**

Thus $f_2(x)$ is a strictly deceasing function for $x \geq 3$.

Since $f_2(3) = 3.375$ and $\lim_{x\to\infty} f_2(x) = e \approx 2.718$, for $x \geq 3$, $3.375 \geq f_2(x) = \left(\frac{x}{x-1}\right)^x \geq e$ **— (2.1.7)**

Since for $x \geq 3$, $f_1(x)$ has a lower bound of 4 and $f_2(x)$ has an upper bound of 3.375,

$f_1(x) = \frac{2(2x-1)}{x-1} > f_2(x) = \left(\frac{x}{x-1}\right)^x$ is proven. **— (2.1.8)**

Lemma 2: For $n \geq 2$ and $\lambda \geq 3$, $\binom{\lambda n}{n} > \frac{\lambda^{\lambda n-\lambda+1}}{n(\lambda-1)^{(\lambda-1)n-\lambda+1}}$ **— (2.2)**

Proof:

When $\lambda \geq 3$ and $n = 2$, $\binom{\lambda n}{n} = \binom{2\lambda}{2} = \frac{2\lambda(2\lambda-1)(2\lambda-2)!}{2(2\lambda-2)!} = \lambda(2\lambda-1)$ **— (2.2.1)**

$\frac{\lambda^{\lambda n-\lambda+1}}{n(\lambda-1)^{(\lambda-1)n-\lambda+1}} = \frac{\lambda^{2\lambda-\lambda+1}}{2(\lambda-1)^{2(\lambda-1)-\lambda+1}} = \frac{\lambda(\lambda-1)}{2} \cdot \left(\frac{\lambda}{\lambda-1}\right)^\lambda$ **— (2.2.2)**

In **(2.1)** when $x = \lambda \geq 3$, we have $\frac{2(2\lambda-1)}{\lambda-1} > \left(\frac{\lambda}{\lambda-1}\right)^\lambda$ **— (2.2.3)**

Since $\frac{\lambda(\lambda-1)}{2}$ is a positive number for $\lambda \geq 3$, referring to **(2.2.1)** and **(2.2.2)**, when $\frac{\lambda(\lambda-1)}{2}$ multiplies to both sides of **(2.2.3)**, we have

$$\left(\frac{\lambda(\lambda-1)}{2}\right)\left(\frac{2(2\lambda-1)}{\lambda-1}\right) = \lambda(2\lambda-1) = \binom{\lambda n}{n} > \left(\frac{\lambda(\lambda-1)}{2}\right)\left(\frac{\lambda}{\lambda-1}\right)^\lambda = \frac{\lambda^{\lambda n-\lambda+1}}{n(\lambda-1)^{(\lambda-1)n-\lambda+1}}$$

Thus, $\binom{\lambda n}{n} > \frac{\lambda^{\lambda n-\lambda+1}}{n(\lambda-1)^{(\lambda-1)n-\lambda+1}}$ when $\lambda \geq 3$ and $n = 2$. **— (2.2.4)**

By induction on n, when $\lambda \geq 3$, if $\binom{\lambda n}{n} > \frac{\lambda^{\lambda n-\lambda+1}}{n(\lambda-1)^{(\lambda-1)n-\lambda+1}}$ is true for n, then for n+1, we have

$$\binom{\lambda(n+1)}{n+1} = \binom{\lambda n+\lambda}{n+1} = \frac{(\lambda n+\lambda)(\lambda n+\lambda-1)\cdots(\lambda n+2)(\lambda n+1)}{(\lambda n+\lambda-n-1)(\lambda n+\lambda-n-2)\cdots(\lambda n-n+1)(n+1)} \cdot \binom{\lambda n}{n}$$

$$\binom{\lambda(n+1)}{n+1} > \frac{(\lambda n+\lambda)(\lambda n+\lambda-1)\cdots(\lambda n+2)(\lambda n+1)}{(\lambda n+\lambda-n-1)(\lambda n+\lambda-n-2)\cdots(\lambda n-n+1)(n+1)} \cdot \frac{\lambda^{\lambda n-\lambda+1}}{n(\lambda-1)^{(\lambda-1)n-\lambda+1}}$$

$$\binom{\lambda(n+1)}{n+1} > \frac{(\lambda n+\lambda)(\lambda n+\lambda-1)\cdots(\lambda n+2)}{(\lambda n+\lambda-n-1)(\lambda n+\lambda-n-2)\cdots(\lambda n-n+1)} \cdot \frac{\lambda n+1}{n} \cdot \frac{1}{(n+1)} \cdot \frac{\lambda^{\lambda n-\lambda+1}}{(\lambda-1)^{(\lambda-1)n-\lambda+1}}$$

Notice $\frac{\lambda n+1}{n} > \lambda$, and $\frac{(\lambda n+\lambda)(\lambda n+\lambda-1)\cdots(\lambda n+2)}{(\lambda n+\lambda-n-1)(\lambda n+\lambda-n-2)\cdots(\lambda n-n+1)} > (\frac{\lambda}{\lambda-1})^{(\lambda-1)}$

because $\frac{\lambda n+\lambda}{\lambda n+\lambda-n-1} = \frac{\lambda}{\lambda-1}$; $\frac{\lambda n+\lambda-1}{\lambda n+\lambda-n-2} > \frac{\lambda}{\lambda-1}$; $\cdots$ $\frac{\lambda n+2}{\lambda n-n+1} > \frac{\lambda}{\lambda-1}$.

Thus $\binom{\lambda(n+1)}{n+1} > \frac{\lambda^{\lambda-1}}{(\lambda-1)^{(\lambda-1)}} \cdot \frac{\lambda}{1} \cdot \frac{1}{(n+1)} \cdot \frac{\lambda^{\lambda n-\lambda+1}}{(\lambda-1)^{(\lambda-1)n-\lambda+1}} = \frac{\lambda^{\lambda(n+1)-\lambda+1}}{(n+1)(\lambda-1)^{(\lambda-1)(n+1)-\lambda+1}}$ **— (2.2.5)**

From **(2.2.4)** and **(2.2.5)**, we have for $n \geq 2$ and $\lambda \geq 3$, $\binom{\lambda n}{n} > \frac{\lambda^{\lambda n-\lambda+1}}{n(\lambda-1)^{(\lambda-1)n-\lambda+1}}$

Thus, **Lemma 2** is proven.

3. A Prime Number between $(\lambda-1)n$ and λn when $n \geq (\lambda-2) \geq 24$

Proposition:

For $n \geq (\lambda-2) \geq 24$, there exists at least a prime number p such that $(\lambda-1)n < p \leq \lambda n$. **— (3.1)**

Proof:

When $n \geq (\lambda-2) \geq 24$, in $\Gamma_{\lambda n \geq p > n}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\}$, $p \geq n+1 > \sqrt{(n+2)n} > \lfloor\sqrt{\lambda n}\rfloor$. Referring to **(1.5)**, we have $0 \leq v_p\left(\Gamma_{\lambda n \geq p > n}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\}\right) \leq R(p) \leq 1$. And $n \geq (\lambda-2) \geq \lfloor\sqrt{\lambda n}\rfloor \geq 24$.

$\Gamma_{\lambda n \geq p > n}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\} =$

$= \Gamma_{\lambda n \geq p > (\lambda-1)n}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\} \cdot \prod_{i=1}^{i=\lambda-2}\left(\Gamma_{\frac{(\lambda-1)n}{i} \geq p > \frac{\lambda n}{i+1}}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\} \cdot \Gamma_{\frac{\lambda n}{i+1} \geq p > \frac{(\lambda-1)n}{i+1}}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\}\right)$

In $\prod_{i=1}^{\lambda-2}\left(\Gamma_{\frac{(\lambda-1)n}{i} \geq p > \frac{\lambda n}{i+1}}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\}\right)$, for every distinct prime number p in these ranges, the numerator $(\lambda n)!$ has the product of $p \cdot 2p \cdot 3p \ldots ip = (i)! \cdot p^i$. The denominator $((\lambda-1)n)!$ also has the same product of $(i)! \cdot p^i$. Thus, they cancel to each other in $\frac{(\lambda n)!}{((\lambda-1)n)!}$.

Referring to **(1.2)**, $\prod_{i=1}^{\lambda-2}\left(\Gamma_{\frac{(\lambda-1)n}{i} \geq p > \frac{\lambda n}{i+1}}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\}\right) = 1$.

Therefore, when $n \geq \lambda-2 \geq 24$,

$\Gamma_{\lambda n \geq p > n}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\} = \Gamma_{\lambda n \geq p > (\lambda-1)n}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\} \cdot \prod_{i=1}^{i=\lambda-2}\left(\Gamma_{\frac{\lambda n}{i+1} \geq p > \frac{(\lambda-1)n}{i+1}}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\}\right) =$

$= \prod_{i=1}^{i=\lambda-1}\left(\Gamma_{\frac{\lambda n}{i} \geq p > \frac{(\lambda-1)n}{i}}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\}\right)$. **— (3.2)**

Referring to **(1.7)**, $\binom{\lambda n}{n} < \Gamma_{\lambda n \geq p > n}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\} \cdot 2^{2n-3} \cdot (\lambda n)^{\frac{\sqrt{\lambda n}}{3}+1}$. Applying this inequality to **(2.2)**, when $n \geq (\lambda-2) \geq 24$, we have

$\frac{\lambda^{\lambda n-\lambda+1}}{n(\lambda-1)^{(\lambda-1)n-\lambda+1}} < \binom{\lambda n}{n} < \Gamma_{\lambda n \geq p > n}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\} \cdot 2^{2n-3} \cdot (\lambda n)^{\frac{\sqrt{\lambda n}}{3}+1}$.

$\frac{\lambda^{\lambda n-\lambda+1}}{n(\lambda-1)^{(\lambda-1)n-\lambda+1}} < \Gamma_{\lambda n \geq p > n}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\} \cdot 2^{2n-3} \cdot (\lambda n)^{\frac{\sqrt{\lambda n}}{3}+1}$. Since $(\lambda n)^{\frac{\sqrt{\lambda n}}{3}+1} > 1$ and $2^{2n-3} > 1$,

$$\Gamma_{\lambda n\geq p>n}\{\tfrac{(\lambda n)!}{((\lambda-1)n)!}\} > \frac{\lambda^{\lambda n-\lambda+1}}{(\lambda n)^{\frac{\sqrt{\lambda n}}{3}+1}\cdot 2^{2n-3}\cdot n(\lambda-1)^{(\lambda-1)n-\lambda+1}} = \frac{2\lambda^2\cdot\left(\left(\frac{\lambda-1}{4}\right)\cdot\left(\frac{\lambda}{\lambda-1}\right)^{\lambda}\right)^{(n-1)}}{(\lambda n)^{\frac{\sqrt{\lambda n}}{3}+2}}$$

Referring to **(2.1.7)**, when $\lambda \geq 3$, $\left(\frac{\lambda}{\lambda-1}\right)^{\lambda} \geq e$,

thus, $\Gamma_{\lambda n\geq p>n}\{\tfrac{(\lambda n)!}{((\lambda-1)n)!}\} > \frac{2\lambda^2\cdot\left(\left(\frac{\lambda-1}{4}\right)\cdot\left(\frac{\lambda}{\lambda-1}\right)^{\lambda}\right)^{(n-1)}}{(\lambda n)^{\frac{\sqrt{\lambda n}}{3}+2}} \geq \frac{2\lambda^2\cdot\left(\left(\frac{\lambda-1}{4}\right)\cdot e\right)^{(n-1)}}{(\lambda n)^{\frac{\sqrt{\lambda n}}{3}+2}} = f_3(n,\lambda)$ **— (3.3)**

Let $f_3(x,y) = \frac{2y^2\cdot\left(\left(\frac{y-1}{4}\right)\cdot e\right)^{(x-1)}}{(xy)^{\frac{\sqrt{xy}}{3}+2}}$ where both x and y are positive real numbers. **— (3.4)**

When $x = (y-2)$, $f_3(x,y) = \frac{2(x+2)^2\cdot\left(\left(\frac{x+1}{4}\right)\cdot e\right)^{(x-1)}}{(x\cdot(x+2))^{\frac{\sqrt{x\cdot(x+2)}}{3}+2}} > f_4(x) = \frac{2(x+2)^2\cdot\left(\left(\frac{x+1}{4}\right)\cdot e\right)^{(x-1)}}{(x\cdot(x+2))^{\frac{x+1}{3}+2}}$ **— (3.5)**

$f_4{}'(x) = f_4(x)\cdot\left(\frac{2}{x+2} + ln\left(\frac{x+1}{4}\right) + \frac{4}{3} - \frac{2}{x+1} - \frac{1}{3}ln(x\cdot(x+2)) - \frac{7}{3x} - \frac{5}{3(x+2)}\right) = f_4(x)\cdot f_5(x)$

where $f_5(x) = \frac{2}{x+2} + ln\left(\frac{x+1}{4}\right) + \frac{4}{3} - \frac{2}{x+1} - \frac{1}{3}ln(x\cdot(x+2)) - \frac{7}{3x} - \frac{5}{3(x+2)}$

$f_5{}'(x) = \frac{4x+6}{(x+1)^2\cdot(x+2)^2} + \frac{x^2+2x-2}{3x(x+1)(x+2)} + \frac{7}{3x^2} + \frac{5}{3(x+2)^2} > 0$. Thus, $f_5(x)$ is a strictly increasing function for $x \geq 1$.

When $x = 7$, $f_5(x) = \frac{2}{7+2} + ln\left(\frac{7+1}{4}\right) + \frac{4}{3} - \frac{2}{7+1} - \frac{1}{3}ln(7) - \frac{1}{3}ln(7+2) - \frac{7}{21} - \frac{5}{27} > 0$.

Thus, for $x \geq 7$, $f_5(x) > 0$. Then, $f_4{}'(x) = f_4(x)\cdot f_5(x) > 0$. Thus, $f_4(x)$ is a strictly increasing function for $x \geq 7$.

When $x = 16$, $f_4(x) = \frac{2\cdot(18)^2\cdot\left(\frac{17}{4}\right)^{15}\cdot e^{15}}{(16\cdot 18)^{\frac{16+1}{3}+2}} \approx \frac{5.647\text{E}+18}{7.167\text{E}+18} < 1$, then for $x \leq 16$, $f_4(x) < 1$. **— (3.6)**

When $x = 17$, $f_4(x) = \frac{2\cdot(19)^2\cdot\left(\frac{18}{4}\right)^{16}\cdot e^{16}}{(17\cdot 19)^{\frac{17+1}{3}+2}} \approx \frac{1.814\text{E}+20}{1.185\text{E}+20} > 1$, then for $x \geq 17$, $f_4(x) > 1$. **— (3.7)**

Referring to **(3.5)**, when $x = (y-2) \geq 17$, $f_3(x,y) > f_4(x) > 1$. **— (3.8)**

From **(3.4)**, $f_3(x,y) = \frac{2y^2\cdot\left(\left(\frac{y-1}{4}\right)\cdot e\right)^{(x-1)}}{(xy)^{\frac{\sqrt{xy}}{3}+2}}$

$\frac{\partial f_3(x,y)}{\partial x} = f_3(x,y)\cdot\left(ln\left(\frac{y-1}{4}\right) + 1 - \frac{\sqrt{y}}{6\sqrt{x}}\cdot ln(yx) - \frac{\sqrt{y}}{3\sqrt{x}} - \frac{2}{x}\right) = f_3(x,y)\cdot f_6(x,y)$ **— (3.9)**

where $f_6(x,y) = ln\left(\frac{y-1}{4}\right) + 1 - \frac{\sqrt{y}}{6\sqrt{x}}\cdot ln(yx) - \frac{\sqrt{y}}{3\sqrt{x}} - \frac{2}{x}$

$\frac{\partial f_6(x,y)}{\partial x} = \frac{\sqrt{y}}{12x\sqrt{x}}\cdot ln(y) + \frac{\sqrt{y}}{12x\sqrt{x}}\cdot ln(x) + \frac{\sqrt{y}}{6x\sqrt{x}} + \frac{\sqrt{y}}{6x\sqrt{x}} + \frac{2}{x^2} > 0$ when $x \geq 1$ and $y \geq 1$.

Thus, $f_6(x,y)$ is a strictly increasing function with respect to x when $x \geq 1$ and $y \geq 1$.

When $x = (y-2) \geq 7$, $f_6(x,y) = ln\left(\frac{9-1}{4}\right) + 1 - \frac{\sqrt{9}}{6\sqrt{7}} \cdot ln(7 \cdot 9) - \frac{\sqrt{9}}{3\sqrt{7}} - \frac{2}{7} > 0$.

Thus, when $x \geq (y-2) \geq 7$, $f_6(x,y) > 0$, then from **(3.9)**, $\frac{\partial f_3(x,y)}{\partial x} = f_3(x,y) \cdot f_6(x,y) > 0$.

Thus, $f_3(x,y)$ is a strictly increasing function with respect to x when $x \geq (y-2) \geq 7$. **— (3.10)**

Referring to **(3.8)**, when $x = (y-2) \geq 17$, $f_3(x,y) > 1$. Thus, when $x \geq (y-2) \geq 17$, $f_3(x,y) > 1$.

Let $x = n$ and $y = \lambda$, then when $n \geq (\lambda-2) \geq 17$, $f_3(n,\lambda) > 1$. **— (3.11)**

From **(3.2)**, **(3.3)** and **(3.11)**, when $n \geq (\lambda-2) \geq 24$,

$\Gamma_{\lambda n \geq p > n}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\} = \prod_{i=1}^{i=\lambda-1}\left(\Gamma_{\frac{\lambda n}{i} \geq p > \frac{(\lambda-1)n}{i}}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\}\right) > f_3(n,\lambda) > 1$. **— (3.12)**

Referring to **(3.10)**, $\prod_{i=1}^{i=\lambda-1}\left(\Gamma_{\frac{\lambda n}{i} \geq p > \frac{(\lambda-1)n}{i}}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\}\right)$ is a strictly increasing function with respect to n when $n \geq (\lambda-2) \geq 24$. **— (3.13)**

$\prod_{i=1}^{i=\lambda-1}\left(\Gamma_{\frac{\lambda n}{i} \geq p > \frac{(\lambda-1)n}{i}}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\}\right) = \Gamma_{\lambda n \geq p > (\lambda-1)n}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\} \cdot \prod_{i=1}^{i=\lambda-2}\left(\Gamma_{\frac{\lambda n}{i+1} \geq p > \frac{(\lambda-1)n}{i+1}}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\}\right)$

Referring to **(1.1)**, $\Gamma_{\lambda n \geq p > (\lambda-1)n}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\} \geq 1$ and $\prod_{i=1}^{i=\lambda-2}\left(\Gamma_{\frac{\lambda n}{i+1} \geq p > \frac{(\lambda-1)n}{i+1}}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\}\right) \geq 1$.

Referring to **(3.12)**, at least one of these two parts is greater than 1.

If $\Gamma_{\lambda n \geq p > (\lambda-1)n}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\} > 1$, then referring to **(1.3)**, there exists at least a prime number p such that $(\lambda-1)n < p \leq \lambda n$. **— (3.14)**

If $\prod_{i=1}^{i=\lambda-2}\left(\Gamma_{\frac{\lambda n}{i+1} \geq p > \frac{(\lambda-1)n}{i+1}}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\}\right) = 1$, then $\Gamma_{\lambda n \geq p > (\lambda-1)n}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\} > 1$. **— (3.15)**

If $\prod_{i=1}^{i=\lambda-2}\left(\Gamma_{\frac{\lambda n}{i+1} \geq p > \frac{(\lambda-1)n}{i+1}}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\}\right) > 1$, then at least one factor $\Gamma_{\frac{\lambda n}{i+1} \geq p > \frac{(\lambda-1)n}{i+1}}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\} > 1$.

When a factor $\Gamma_{\frac{\lambda n}{i+1} \geq p > \frac{(\lambda-1)n}{i+1}}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\} > 1$, let integer $m = (i+1)n$, then $m > n$.

Referring to **(3.13)**, $\Gamma_{\frac{\lambda m}{i+1} \geq p > \frac{(\lambda-1)m}{i+1}}\{\frac{(\lambda m)!}{((\lambda-1)m)!}\} = \Gamma_{\lambda n \geq p > (\lambda-1)n}\{\frac{(\lambda m)!}{((\lambda-1)m)!}\} > 1$. **— (3.16)**

$\frac{(\lambda n)!}{((\lambda-1)n)!} = (\lambda n)\cdot(\lambda n-1)\cdot(\lambda n-2)\cdots((\lambda-1)n+1)$. This product has n factors.

$\frac{(\lambda m)!}{((\lambda-1)m)!} = (\lambda m)\cdot(\lambda m-1)\cdot(\lambda m-2)\cdots((\lambda-1)m+1)$. This product has $m= (i+1)n$ factors.

Since $m = (i+1)n$, $\frac{(\lambda m)!}{((\lambda-1)m)!}$ contains the factors of $\frac{(\lambda n)!}{((\lambda-1)n)!} \cdot (i+1)^n$.

$\frac{(\lambda m)!}{((\lambda-1)m)!} =$

$= \Gamma_{\lambda m \geq p > \lambda n}\{\frac{(\lambda m)!}{((\lambda-1)m)!}\} \cdot \Gamma_{\lambda n \geq p > (\lambda-1)n}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\} \cdot \Gamma_{\lambda n \geq p > (\lambda-1)n}\{(i+1)^n\} \cdot \Gamma_{(\lambda-1)n \geq p}\{\frac{(\lambda m)!}{((\lambda-1)m)!}\}$.

Notice that $(i+1) \leq (\lambda-1)$. Prime numbers in $(i+1)$ are not in the range of $\lambda n \geq p > (\lambda-1)n$.

Since prime numbers in $\Gamma_{\lambda m \geq p > \lambda n}\{\frac{(\lambda m)!}{((\lambda-1)m)!}\}$, $\Gamma_{\lambda n \geq p > (\lambda-1)n}\{(i+1)^n\}$, and $\Gamma_{(\lambda-1)n \geq p}\{\frac{(\lambda m)!}{((\lambda-1)m)!}\}$ are not in the range of $\lambda n \geq p > (\lambda-1)n$, $\Gamma_{\lambda n \geq p > (\lambda-1)n}\{\ \Gamma_{\lambda m \geq p > \lambda n}\{\frac{(\lambda m)!}{((\lambda-1)m)!}\}\ \} = 1$,

$\Gamma_{\lambda n \geq p > (\lambda-1)n}\{\ (i+1)^n\ \} = 1$, and $\Gamma_{\lambda n \geq p > (\lambda-1)n}\{\ \Gamma_{(\lambda-1)n \geq p}\{\frac{(\lambda m)!}{((\lambda-1)m)!}\}\ \} = 1$.

Referring to **(3.16)**, $\Gamma_{\lambda n \geq p > (\lambda-1)n}\{\frac{(\lambda m)!}{((\lambda-1)m)!}\} = \Gamma_{\lambda n \geq p > (\lambda-1)n}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\} > 1$.

Thus, If $\prod_{i=1}^{i=\lambda-2}\left(\Gamma_{\frac{\lambda n}{i+1} \geq p > \frac{(\lambda-1)n}{i+1}}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\}\right) > 1$, then $\Gamma_{\lambda n \geq p > (\lambda-1)n}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\} > 1$, and there exists at least a prime number p such that $(\lambda-1)n < p \leq \lambda n$. **— (3.17)**

Combining **(3.13)**, **(3.14)**, **(3.15)**, and **(3.17)**, we have proven the **Proposition**, **(3.1)**:

For $n \geq (\lambda - 2) \geq 24$, there exists at least a prime number p such that $(\lambda - 1)n < p \leq \lambda n$.

It becomes a theorem: **Theorem (3.1)**.

4. The Proof of Legendre's Conjecture

Legendre's Conjecture states that there is a prime number between n^2 and $(n+1)^2$ for every positive integer n. **— (4.1)**

Proof:

Referring to **Theorem (3.1)**, for integers $j \geq k - 2 \geq 24$, there exists at least a prime number p such that $j(k-1) < p \leq jk$. **— (4.2)**

When $k = j + 1 \geq 26$, then $j = k - 1 \geq 25$

Applying $k = j + 1$ into **(4.2)**, then $j^2 < p \leq j(j+1) < (j+1)^2$

Let $n = j \geq 25$, then we have $n^2 < p < (n+1)^2$. **— (4.3)**

For $1 \leq n \leq 24$, we have a table, **Table 1**, that shows Legendre's conjecture valid. **— (4.4)**

Table 1: For $1 \leq n \leq 24$, there is a prime number between n^2 and $(n+1)^2$.

n	1	2	3	4	5	6	7	8	9	10	11	12
n^2	1	4	9	16	25	36	49	64	81	100	121	144
p	3	5	11	19	29	41	53	67	83	103	127	149
$(n+1)^2$	4	9	16	25	36	49	64	81	100	121	144	169
n	13	14	15	16	17	18	19	20	21	22	23	24
n^2	169	196	225	256	289	324	361	400	441	484	529	576
p	173	199	229	263	307	331	373	409	449	491	541	587
$(n+1)^2$	196	225	256	289	324	361	400	441	484	529	576	625

Combining **(4.3)** and **(4.4)**, we have proven Legendre's conjecture.

Extension of Legendre's conjecture

There are at least two prime numbers, p_n and p_m , between j^2 and $(j+1)^2$ for every positive integer j such that $j^2 < p_n \le j(j+1)$ and $j(j+1) < p_m < (j+1)^2$ where p_n is the n^{th} prime number, p_m is the m^{th} prime number, and $m \ge n + 1$. **— (4.5)**

Proof:

Referring to **Theorem (3.1)**, for integers $j \ge k-2 \ge 24$, there exists at least a prime number p such that $j(k-1) < p \le jk$.

When $k-1 = j \ge 25$, then $j(k-1) = j^2 < p_n \le jk = j(j+1)$. Thus, there is at least a prime number p_n such that $j^2 < p_n \le j(j+1)$ when $j = k-1 \ge 25$.

When $j = k-2 \ge 25$, then $k = j + 2$. Thus, $j(k-1) = j(j+1) < p_m \le jk = j(j+2) < (j+1)^2$. Thus, there is at least another prime number p_m such that $j(j+1) < p_m < (j+1)^2$ when $j = k-2 \ge 25$.

Thus, when $j \ge 25$, there are at least two prime numbers p_n and p_m between j^2 and $(j+1)^2$ such that $j^2 < p_n \le j(j+1) < p_m < (j+1)^2$ where $m \ge n + 1$ for $p_m > p_n$. **— (4.6)**

For $1 \le j \le 24$, we have a table, **Table 2**, that shows **(4.5)** valid. **— (4.7)**

Table 2: For $1 \le j \le 24$, there are 2 prime numbers such that $j^2 < p_n \le j(j+1) < p_m < (j+1)^2$.

j	1	2	3	4	5	6	7	8	9	10	11	12
j^2	1	4	9	16	25	36	49	64	81	100	121	144
p_n	2	5	11	19	29	41	53	67	83	103	127	149
$j(j+1)$	2	6	12	20	30	42	56	72	90	110	132	156
p_m	3	7	13	23	31	43	59	73	97	113	137	163
$(j+1)^2$	4	9	16	25	36	49	64	81	100	121	144	169
j	13	14	15	16	17	18	19	20	21	22	23	24
j^2	169	196	225	256	289	324	361	400	441	484	529	576
p_n	173	199	229	263	393	331	373	409	449	491	541	587
$j(j+1)$	182	210	240	272	306	342	380	420	462	506	552	600
p_m	191	211	251	277	311	349	389	431	467	521	557	613
$(j+1)^2$	196	225	256	289	324	361	400	441	484	529	576	625

Combining **(4.6)** and **(4.7)**, we have proven **(4.5)**. It becomes a theorem: **Theorem (4.5)**.

5. The Proofs of Three Related Conjectures

Oppermann's conjecture was proposed by Ludvig Oppermann [4] in March 1877. It states that for every integer $x > 1$, there is at least one prime number between $x(x-1)$ and x^2, and at least another prime between x^2 and $x(x+1)$. **— (5.1)**

Proof:

Theorem (4.5) states there are at least two prime numbers, p_n and p_m, between j^2 and $(j+1)^2$ for every positive integer j such that $j^2 < p_n \leq j(j+1)$ and $j(j+1) < p_m < (j+1)^2$ where $m \geq n+1$ for $p_m > p_n$.

$j(j+1)$ is a composite number except $j = 1$. Since $j^2 < p_n \leq j(j+1)$ is valid for every positive integer j, when we replace j with $j+1$, we have $(j+1)^2 < p_v < (j+1)(j+2)$.

Thus, we have $j(j+1) < p_m < (j+1)^2 < p_v < (j+1)(j+2)$. **— (5.2)**

When $x > 1$, then $(x-1) \geq 1$. Substitute j with $(x-1)$ in **(5.2)**, we have

$x(x-1) < p_m < x^2 < p_v < x(x+1)$ **— (5.3)**

Thus, we have proven Oppermann's conjecture.

Brocard's conjecture is named after Henri Brocard [5]. It states that there are at least 4 prime numbers between $(p_n)^2$ and $(p_{n+1})^2$, where p_n is the n^{th} prime number, for every $n > 1$. **— (5.4)**

Proof:

Theorem (4.5) states there are at least two prime numbers, p_n and p_m, between j^2 and $(j+1)^2$ for every positive integer j such that $j^2 < p_n \leq j(j+1)$ and $j(j+1) < p_m < (j+1)^2$ where $m \geq n+1$ for $p_m > p_n$. When $j > 1$, $j(j+1)$ is a composite number. Then **Theorem (4.5)** can be written as $j^2 < p_n < j(j+1)$ and $j(j+1) < p_m < (j+1)^2$.

In the series of prime numbers: p_1=2, p_2=3, p_3=5, p_4=7, p_5=11... all prime numbers except p_1 are odd numbers. Their gaps are two or more. Thus when $n > 1$, $(p_{n+1} - p_n) \geq 2$.

Thus, we have $p_n < (p_n + 1) < p_{n+1}$ when $n > 1$. **— (5.5)**

Applying **Theorem (4.5)** to **(5.5)**, when $n > 1$, we have at least two prime numbers p_{m1}, p_{m2} in between $(p_n)^2$ and $(p_n+1)^2$ such that $(p_n)^2 < p_{m1} < p_n(p_n+1) < p_{m2} < (p_n+1)^2$, and at least two more prime numbers p_{m3}, p_{m4} in between $(p_n+1)^2$ and $(p_{n+1})^2$ such that $(p_n+1)^2 < p_{m3} < p_{n+1}(p_n+1) < p_{m4} < (p_{n+1})^2$.

Thus, there are at least 4 prime numbers between $(p_n)^2$ and $(p_{n+1})^2$ for $n > 1$ such that

$(p_n)^2 < p_{m1} < p_n(p_n+1) < p_{m2} < (p_n+1)^2 < p_{m3} < p_{n+1}(p_n+1) < p_{m4} < (p_{n+1})^2$ **— (5.6)**

Thus, Brocard's conjecture is proven.

Andrica's conjecture is named after Dorin Andrica [6]. It is a conjecture regarding the gaps between prime numbers. The conjecture states that the inequality $\sqrt{p_{n+1}} - \sqrt{p_n} < 1$ holds for all n where p_n is the n^{th} prime number. If $g_n = p_{n+1} - p_n$ denotes the n^{th} prime gap, then Andrica's conjecture can also be rewritten as $g_n < 2\sqrt{p_n} + 1$. **— (5.7)**

Proof:

From **Theorem (4.5)**, for every positive integer j, there are at least two prime numbers p_n and p_m between j^2 and $(j+1)^2$ such that $j^2 < p_n \leq j(j+1) < p_m < (j+1)^2$ where $m \geq n+1$ for $p_m > p_n$.

Since $m \geq n+1$, we have $p_m \geq p_{n+1}$.

Thus, we have $j^2 < p_n$. **— (5.8)**

And $p_{n+1} \leq p_m < (j+1)^2$. **— (5.9)**

Since j, p_n , p_{n+1} and $(j+1)$ are positive integers,
$j < \sqrt{p_n}$ **— (5.10)**

And $\sqrt{p_{n+1}} < j + 1$ **— (5.11)**

Applying **(5.10)** to **(5.11)**, we have $\sqrt{p_{n+1}} < \sqrt{p_n} + 1$. **— (5.12)**

Thus, $\sqrt{p_{n+1}} - \sqrt{p_n} < 1$ holds for all n since in **Theorem (4.5)**, j holds for all positive integers.

Using the prime gap to prove the conjecture, from **(5.8)** and **(5.9)**, we have
$g_n = p_{n+1} - p_n < (j+1)^2 - j^2 = 2j + 1$. From **(5.10)**, $j < \sqrt{p_n}$.
Thus, $g_n = p_{n+1} - p_n < 2\sqrt{p_n} + 1$. **— (5.13)**

Thus, Andrica's conjecture is proven.

6. References

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[6] *Wikipedia,* https://en.wikipedia.org/wiki/Andrica%27s_conjecture

[7] Wikipedia, https://en.wikipedia.org/wiki/Proof_of_Bertrand%27s_postulate, Lemma 4.