# Validity of the Collatz Conjecture

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**Abstract**

We establish an equivalent condition to the validity of the Collatz conjecture: The Collatz conjecture is true if and only if every natural number n $\geq$ 3 can be represented in the form

$$n = (2^m - 2^{m-2} b_{m-1} - \sum_{i=1}^{m-2} b_i\, 2^{i-1} 3^{\sum_{j=i+1}^{m-1} b_j})/ 3^{\sum_{j=1}^{m-1} b_j}$$

for some m $\in N$ , $b_1, b_2, \ldots\ldots b_{m-1} \in \{1, 0\}$.

## 1. Introduction

Denote by $N = \{1,2,3\ldots\ldots\ldots\ldots\}$ the set of all natural numbers, define a recursive equation introduced by R. Terras[1] :

$$a_{n+1} = (3^{b_n} a_n + b_n)/2 \tag{1}$$

where $b_n$ = 1 when $a_n$ is odd and $b_n$ = 0 when $a_n$ is even. The Collatz conjecture [2] asserts that for every positive integer $a_1$ and by applying eq.(1) , there exists k $\in$ N such that $a_k = 1$.

For example,

Let $a_1 = 2$ then $a_2 = 1$;

$a_1 = 3$ then $a_2 = 5$, $a_3 = 8$, $a_4 = 4$, $a_5 = 2$, $a_6 = 1$;

$a_1 = 16$ then $a_2 = 8$, $a_3 = 4$, $a_4 = 2$, $a_5 = 1$.

## 2. The collatz tree

Define the iterating function by

$$T(n) = (2n - b_n)/3^{b_n} , \tag{2}$$

where $b_n$ = 0 or $b_n$ = 0 and $b_n$ = 1 if n ≡ 2 mod 3. Thus, denote n as a node value it can have one branch or two branches coming out.

Let G be a tree with nodes represented by integers generated from eq.(2) starting with n =1. This tree is called the Collatz tree which is divided into many levels as shown in Figure 1.

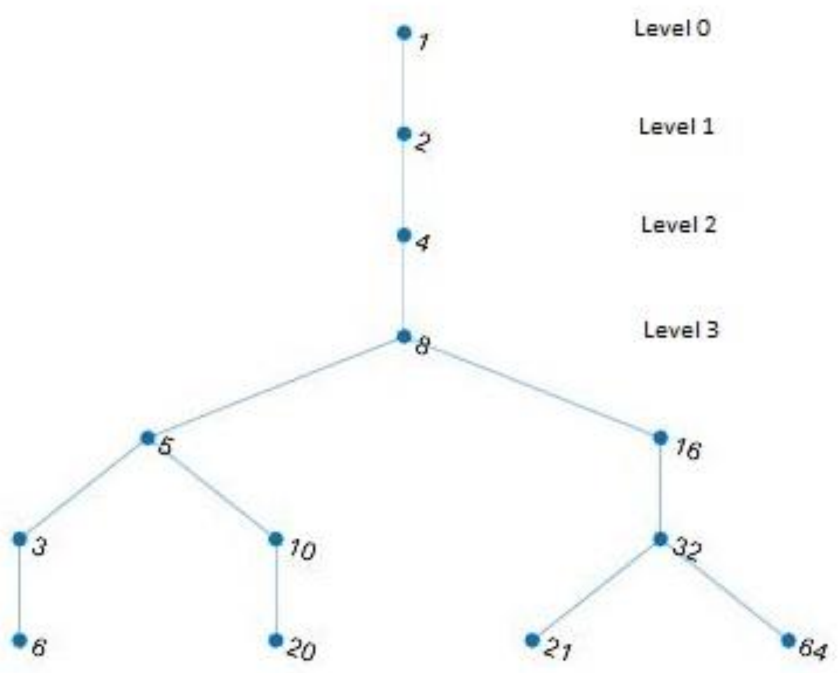


Figure 1. Six levels of the Collatz tree

## 3. Representation of natural numbers

Assume the Collatz conjecture is true; by starting with $a_1 \geq 3$ and takes m steps following eq.(1) to reach 1, thus $a_m = 2$, $a_{m+1} = 1$. All equations are given by

$$3^{b_n} a_n - 2a_{n+1} = -b_n \tag{3}$$

where n = 1,2,……., (m-2), $b_n \in \{1, 0\}$. and

$$3^{b_{m-1}} a_{m-1} = 4 - b_{m-1} \tag{4}$$

By substitute eq. (4) back into eq.(3), finally we have

$$a_1 = (2^m - 2^{m-2} b_{m-1} - \sum_{i=1}^{m-2} b_i\, 2^{i-1} 3^{\sum_{j=i+1}^{m-1} b_j})/\, 3^{\sum_{j=1}^{m-1} b_j} \tag{5}$$

where $b_1, b_2, \ldots\ldots b_{m-1} \in \{1, 0\}$.

For example,

If $a_1 = 5$ then m = 4, $b_1 = 1,\ b_2 = 0,\ b_3 = 0;$

$a_1 = 3$ then m = 5, $b_1 = 1,\ b_2 = 1,\ b_3 = 0.\ b_4 = 0;$

$a_1 = 7$ then m = 11, $b_1 = 1,\ b_2 = 1,\ b_3 = 1.\ b_4 = 0,$

$$b_5 = 1,\ b_6 = 0,\ b_7 = 0.\ b_8 = 1,\ b_9 = 0.\ b_{10} = 0.$$

$a_1 = 1$ and $a_1 = 2$ are trivial cases.

## 4. Conclusion

For any $a_1 \geq 3$ finding m and $b_1, b_2, \ldots\ldots b_{m-1} \in \{1, 0\}$ by eq. (5) is hard. Proving the Collatz conjecture seem to be a NP-problem.

## References


[1] R . Terras, (1976). " A stopping time problem on the positive integers". Acta Arithmetica, 30(3), 241-252.

[2] D. Dominici, " Working with 2s and 3s ": eprint: arxiv: 0704.1057v1 [math. DS] 9 April 2007.