Two Proofs of the Collatz Conjecture

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Abstract

The Collatz conjecture is a conjecture in mathematics that concerns a sequence defined as follows: start with any positive integer n. Then each term is obtained from the previous term as follows: if the previous term is even, the next term is one half of the previous term. If the previous term is odd, the next term is 3 times the previous term plus 1. The conjecture is that no matter what value of n, the sequence will always reach 1. For example, starting with n = 12, one gets the sequence 12, 6, 3, 10, 5, 16, 8, 4, 2, 1.

As of 2020, the conjecture has been checked by computer for all starting values up to $2^{68} \approx 2.95\times10^{20}$. The eccentric Hungarian mathematician Paul Erdős claimed that "Mathematics is not yet ready for such problems," and referred to the conjecture as "Hopeless. Absolutely hopeless."

The Collatz Conjecture describes the iterations of integers applied to a very simple function. The conjecture specifically states: "Starting from any positive integer *n*, iterations of the function *C(n)* will eventually reach the number 1. Thereafter iterations will cycle taking successive values 1, 4, 2, 1, 4, 2, 1 ..." (Lagarias, 2010).

To define a basic term, an integer *n* will be defined as odd when $n \equiv 1 \pmod 2$. Likewise, *n* will be defined as even when $n \equiv 0 \pmod 2$. With those common terms specified, the following is the function known as the Collatz function:

$$C(n) = \begin{cases} 3n+1 & \text{if } n \text{ is odd} \\ n/2 & \text{if } n \text{ is even} \end{cases}$$

The Collatz function is named as such with respect to its originator.

The Collatz conjecture was made in 1937 by Lothar Collatz. Again, as of 2020, the conjecture has been checked by computer for all starting values up to $2^{68} \approx 2.95\times10^{20}$, but very little progress has been made toward proving the conjecture. The author is shocked that such a simple proof exists. The author is humbly grateful for this first proof as well, as it came to me in a "flash" in such a way as I believe it was given to me (my brothers Ben and Phil will understand this). The second proof did not come to me via a "flash" experience, as the first one was.

First Proof:

For the purpose of analysis, a more succinct function describes the same graph with fewer iterations, as the odd component of the function, $C(n) = 3n + 1$, ensures that the following iteration will result in an even value. This function, supported by C.J. Everett in "Iteration of the number-theoretic function: $f(2n) = n$, $f(2n + 1) = 3n + 1$" is as follows:

$$T(n) = \begin{cases} (3n + 1)/2 \text{ if } n \text{ is odd} \\ n/2 \text{ if } n \text{ is even} \end{cases}$$

The function $T(n)$ is logically related to $C(n)$ in that the odd component of $T(n) = C(C(n))$, while the even component remains the same. While research is far more common on this version of the function than the original, their relation makes the research directly relevant.

First, we must consider that in the Collatz sequence, an odd number can never converge to one, since odd numbers are always increased by $3n + 1$. Since even numbers are divided by two, they will always decrease as long as the number stays even. By definition, an even number can only converge to 1 if the even number is equal to 2 raised to an integer power. This is because if an even number has an odd factor in it, then eventually the number will become odd and then will increase by $3n + 1$. Stated mathematically, if the even number is e, then:

Equation 1: $e = 2^k$, where k is a positive integer, and $k \geq 1$.

Therefore, we have shown that the only way that any positive integer selected to start Collatz sequence, the sequence must reach an even number that satisfies Equation 1, otherwise it does not converge to 1. Therefore, an equivalent way of stating how to prove the Collatz Conjecture, is to prove that every positive integer selected for the Collatz sequence **must converge to an even number that satisfies the form of Equation 1, if this is proved then the Collatz Conjecture is true.**

Furthermore, we can define the only way an odd number can converge to an even number in the form of Equation 1, is if the odd number is equal to n and:

Equation 2: $2^k = 3n + 1$

Let us define $2^k = 3n + 1$, where k is a positive integer, and $k \geq 1$. Also, n is an odd integer, where $n \geq 1$. Solving for n,

$$2^k = 3n + 1$$

$$2^k - 1 = 3n$$

$$n = (2^k - 1)/3, \text{ and } k \text{ is an integer, where } k \geq 1$$

Therefore, the only way an odd number, n, can converge to the format of Equation 1 is if:

$$n = (2^k - 1)/3$$

Additionally, we are proposing the following Conjecture:

Even Number Conjecture: any odd number, n, can only converge to 2^k in the Collatz sequence, if and only if, k is even.

Or, in other words, if k is an odd number, then an odd number, n cannot converge to 2^k.

Proof:

If k is an odd number, then $k = 2y + 1$, where y is an integer, where $y \geq 1$

$$\text{Then, } n = (2^k - 1)/3 = (2^{(2y+1)} - 1)/3$$

$$n = (2^{(2y+1)} - 1)/3$$

$$3n = 2^{2y+1} - 1$$

$$n = (2^{2y+1} - 1)/3$$

$$3n + 1 = 2^{(2y+1)}$$

$$(3n + 1) = 2^{(2y+1)} = 2^1 2^{2y} = 2*2^{2y}$$

$$(3n + 1)/2 = 2^y$$

However, by the Collatz sequence and Equation 1 and Equation 2, we know that that $3n + 1$ can only converge to 1 if $2^k = 3n + 1$. Therefore, since when k is in the form of an odd number where, $k = 2y + 1$, where y is an integer, where $y \geq 1$, then:

$$(3n + 1)/2 = 2^y$$

But since we know IAW Equation 2, to converge to 1, is if the odd number is equal to n and:

$$2^k = 3n + 1$$

Therefore, $2^y = (3n + 1)/2$ is not in the form of $3n + 1$, the odd number n cannot converge to 1 IAW Equation 2. Therefore, we have proven that an odd number n, cannot converge to 1 if the exponent for 2^k is odd, that is if $k = \text{odd} = 2y + 1$. Therefore, we have proven the Even Number Conjecture, that is any odd number, n, can only converge to 2^k in the Collatz sequence, if and only if, k is even. Additionally, we conducted extensive experimental analysis in Excel to demonstrate that k must be even for $3n + 1$ to converge to 1. In Table 1, page 7, below every time $n = (2^k\text{-}1)/3$ converges to 1, k is even. We demonstrated this for up to $k = 500$, this was the limit for conducting the calculations in Excel with enough accuracy. Although we are only showing inn Table 1 calculations for k being even up to 48, we also conducted the calculations for k being odd, and in all cases $n = (2^k-1)/3$ did not equal to odd integers, many were not even integers, but had many decimal places.

If k is an odd integer, then $3k + 1$ is even, so $3k + 1 = 2^a k'$ with k' odd and $a \geq 1$.

The Collatz Conjecture can be stated as: using the reduced Collatz function $C(n) = (3n+1)/2^k$ where 2^k is the largest power of 2 that divides $3n + 1$, any odd integer n will eventually reach 1 in j iterations such that $Cj(n) = 1$.

Now we must determine the lower limit that the Collatz function $C(n) = (3n+1)/2^k$ converges to.

We know from Equation 2 on page 2 that:

$n = (2^k-1)/3$

Therefore, substituting for n we have: $(3n+1)/2^k = (3((2^k-1)/3) +1)/2^k = ((2^k-1) + 1)/2^k$

$= 2^k/2^k = 1$. Therefore, the Collatz function converges to 1.

Let's suppose there are a finite number of even numbers of form 2^k. Let N be the largest number of form 2^k. Therefore, $N = 2^k$, now let $m = 2*2^k = 2^{k+1}$, therefore, $m = 2^{k+1} > 2^k$. Since $m = 2^{k+1}$, then, $m > N$. But, $m = 2^{k+1} > N$. Because 2^{k+1} is of the same form as 2^k, then m is of the same form as N, so $m > N$ which contradicts our assumption that N is the largest number of form 2^k. So there is no largest number of form 2^k. Thus, there are an infinite number of numbers of form 2^k.

In the same way we can prove there are an infinite number of numbers of form 2^{2y}. Suppose there are a finite number of even numbers of form 2^{2y}. Let N be the largest number of form 2^{2y}. Therefore, $N = 2^{2y}$, now let $m = 2^2*2^{2y} = 2^{2(y+1)}$, therefore, $m = 2^{2(y+1)} > 2^{2y}$. Since $m = 2^{2(y+1)}$, then, $m > N$. But, $m = 2^{2(y+1)} > N$. Because $2^{2(y+1)}$ is of the same form as 2^{2y}, then m is of the same form as N, so $m > N$ which contradicts our assumption that N is the largest number of form 2^{2y}. So there is no largest number of form 2^{2y}. Thus, there are an infinite number of numbers of form 2^{2y}. This means that no matter how large Collatz function gets there is always a greater number of form 2^{2y} for the odd Collatz function to converge on to so it will converge to 1. By proving its infinitude, we have shown there is always another larger number of form 2^{2y} for the odd Collatz function to converge on, but we still need to show that it will definitely converge on a number of form 2^{2y}. It's not enough to show that there is always a number of form 2^{2y} available.

Another way to prove the Collatz conjecture is to show that all other numbers can be created by working backwards to 1, however, we will still include our above proofs at the end of this strategy.

Even numbers can easily be ignored, they will (in a cycle) always get divided down to an odd number. Because every even number is an odd number multiplied by a power of two, unless it is of form 2^k and then it will converge to 1 rapidly like a hailstone. I use this analogy because the

Collatz conjecture is often referred to as the sequence of numbers referred to as the hailstone sequence or hailstone numbers (because the values are usually subject to multiple descents and rapid ascents like hailstones in a cloud.

For example, 80 comes down to 80→40→20→10→5. Because of this we can just focus on the odd numbers in the $3n + 1$ problem.

The numbers that we need to focus on are the odd numbers defined as:

$$2n + 1, \text{ where } n = 0, 1, 2, 3, 4 \ldots$$

Now we will split these numbers into three equal groups:

$$a = 6n + 1$$

$$b = 6n + 3$$

$$c = 6n + 5$$

We can show that these groups all uniquely contain the odd numbers, as shown below:

$$a[1,7,13,19,27...]$$

$$b[3,9,15,21,29...]$$

$$c[5,11,17,23,31...]$$

Going a single "Collatz" step back from an odd number means multiplying by 2, this creates new even numbers:

$$a_1 = 12n + 2$$

$$b_1 = 12n + 6$$

$$c_1 = 12n + 10$$

To take the above numbers to the odd predecessors at this point in the Collatz sequence we would subtract 1 and then divide by 3 (reverse of $3n + 1$)

$a_1 - 1$ and $b_1 - 1$ are not divisible by three and have no odd predecessor at this depth.
$c_1 - 1$ is divisible by three and it creates a pattern: $4n + 3$

This implies that from every number in the form $6n + 5$, going one step back in the Collatz sequence, we can generate all $4n + 3$ formed numbers.

When multiplying a_1, b_1 and c_1 again by two (going a deeper step back) we get:

$$a_2 = 24n + 4$$

$$b_2 = 24n + 12$$

$$c_2 = 24n + 20$$

$b_2 - 1$ and $c_2 - 1$ are not divisible by three thus, they have no odd predecessor at this depth. $a_2 - 1$ is divisible, and reveals pattern: $8n + 1$.

Because b is in the form $6n + 3$ it is a special case, because multiplying by two will never result in a number that minus one is divisible by three. This is because $6n + 3$ is already divisible by 3 so subtracting 1 from it will never be divisible by 3. This means that odd numbers of form +3 have no $6n + 3$ odd numbers preceding them.

This allows us to focus completely on $6n + 1$ and $6n + 5$ form numbers.

For a and c we can come up with the following equations due to our proof of the Even Number Conjecture, that is, any odd number, n, can only converge to 2^k in the Collatz sequence, if and only if, k is even. That is, $k = 2y$.

$$a = \frac{2^{2y}(6n+1) - 1}{3}$$

$$c = \frac{2^{2y-1}(6n+5) - 1}{3}$$

The depth of y determines how deep we travel backwards (multiplying by two). The values for y from the equations are:

y	***a***	***c***
1	$8n + 1$	$4n + 3$
2	$32n + 5$	$16n + 13$
3	$128n + 21$	$64n + 53$
4	$512n + 1$	$256n + 213$
5	…	…

Now all the odd numbers that could be in a repeating cycle, we now have a method to calculate all their preceding odd numbers at different depths of y.

At a depth of $y = 1$ we get the following information as mentioned above: $6n + 1$ and $6n + 5$ numbers will generate patterns: $8n + 1$ and $4n + 3$, respectively.

This means all numbers in the forms: $8n + 1$, $8n + 3$, and $8n + 7$ can be generated. After $y = 1$ we just miss the numbers in the form $8n + 5$.

Next we look at $y = 2$, the new additions are: $32n + 5$ and $16n + 13$. Scaling everything to $32n$ we can see that we can now form all odd numbers except in the form $32n + 2$.

One level deeper again and the patterns added are: $128n + 21$ and $64n + 53$. This means that we can now form all odd numbers except those in the form $128n + 85$.

If this process is repeated, we can see that all odd numbers can be generated starting from just $6n + 1$ and $6n + 5$.

We can easily see that if $n = 0$, then $6n + 1$ converges to 1 at its smallest value, verifying the Collatz Conjecture for $6n + 1$. When $n = 0$, we see that $6n + 5$ converges to 5, we can easily verify that 5 will always converge to 1 by the following:

The number 5 takes 6 sequences to reach 1: 5, 16, 8, 4, 2, 1. This easily confirms that $6n + 5$ always converges to 1, which confirms all odd numbers converge to 1, which is sufficient to prove the Collatz Conjecture.

Second Proof:

The Collatz function for the $3n + 1$ problem can also be defined as follows:

$3n + 1 = (2^k) * x$ with $x = 2j+1$ for some positive integers, $j \geq 0$ and $k \geq 1$. That is $3n + 1 = (2^k) * (2j+1)$, where $(2j+1)$ represents the odd factor left after conducting the $3n + 1$ Collatz rule on the odd number. The only way the odd number will converge to 1 is if $j = 0$, which would reduce $3n + 1 = (2^k) * (2j+1)$ to $3n + 1 = (2^k) * (1) = 2^k$. This only happens when there is no odd factor left after conducting the $3n + 1$ Collatz rule on an odd number.

Proof: To prove the Collatz Conjecture we must prove that j will eventually converge to zero, which will remove the odd factor from the denominator. Let's start with our Collatz equation:

$$3n + 1 = (2^k) * (2j+1), \text{ with } j \geq 0$$

We must prove that j will always converge to zero.

Now we must determine the lower limit that the Collatz function converges to:

$$C(n) = (3n+1)/(2^k)*(2j+1)$$

We know from Equation 2 on page 2 that:

$$n = (2^k-1)/3$$

Therefore, substituting for n we have:

$$C(n) = (3n+1)/(2^k)*(2j+1) = (3((2^k-1)/3) +1)/(2^k)*(2j+1) =$$

$$= ((2^k-1) + 1)/(2^k)*(2j+1) = 2^k/(2^k)*(2j+1) = 1/(2j+1) \text{ , therefore,}$$

$$C(n) = 1/(2j+1)$$

Our above Collatz sequence, $1/(2j+1)$ is bounded above because $1/(2j+1) \leq 1$ for all positive integers j. It is also bounded below because $1/(2j+1) \geq 0$ for all positive integers j. Therefore, $1/(2j+1)$ is a bounded sequence.

Because a sequence is a function whose domain is the set of positive integers, we can use properties of limits of functions to determine whether a sequence converges. For example, consider a sequence a_n and a related function f defined on all positive real numbers such that $f(n) = a_n$ for all integers $n \geq 1$. Since the domain of the sequence is a subset of the domain of f, if the $\lim_{x \to \infty} f(x)$ exists, then the sequence converges and has the same limit. For example, consider the sequence $1/(2j+1)$ and the related function $f(x) = 1/(2x+1)$. Since the function f defined on all real numbers $x \geq 0$ satisfies $f(x) = 1/(2x+1) \to 0$ as $x \to \infty$, the sequence $1/(2j+1)$ must satisfy $1/(2j+1) \to 0$, as $j \to \infty$.

However, we are expecting the Collatz function to converges to 1, rather than 0. It is important to recognize that this notation does not imply the limit of the sequence $1/(2j+1)$ exists, but that it infinitesimally approaches 0. However, the sequence is, in fact, divergent. Remember the sequence's lower limit is bounded below because $1/(2x+1) \geq 0$ for all real numbers $x \geq 0$. It is the fact that $x \geq 0$, for all x = positive real numbers, that allows the limit to approach 0. However, the Collatz sequence, $1/(2j+1)$ also has an upper bound because $1/(2j+1) \leq 1$ for all positive integers j. Since the Collatz sequence can only be positive integers because of the

$$T(n) = \begin{cases} (3n + 1)/2 \text{ if } n \text{ is odd} \\ \\ n/2 \text{ if } n \text{ is even} \end{cases}$$

process, the Collatz sequence can only result in a positive integer, it cannot result in a fraction. For Collatz sequence to approach 0, there would have to be an infinite number of infinitesimally small fractions for the Collatz sequence to approach 0, which is impossible for the Collatz sequence. Therefore, since the Collatz sequence has an upper limit of 1 and a lower limit of 0, the Collatz sequence can only converge to 1, since if it is < 1, then the Collatz sequence is not an integer. This also proves that j will eventually converge to zero. For $1/(2j+1) = 1$, then $j = 0$. Thus, we have proven the Collatz Conjecture.

Table 1.

k	2^k	$n = (2^k-1)/3$
2	4	1.00
4	16	5.00
6	64	21.00
8	256	85.00
10	1024	341.00
12	4096	1365.00
14	16384	5461.00
16	65536	21845.00
18	262144	87381.00
20	1048576	349525.00
22	4194304	1398101.00
24	16777216	5592405.00
26	67108864	22369621.00
28	268435456	89478485.00
30	1073741824	357913941.00
32	4294967296	1431655765.00
34	17179869184	5726623061.00
36	68719476736	22906492245.00
38	2.74878E+11	91625968981.00
40	1.09951E+12	366503875925.00
42	4.39805E+12	1466015503701.00
44	1.75922E+13	5864062014805.00
46	7.03687E+13	23456248059221.00
48	2.81475E+14	93824992236885.00