Can von Neumann's theory be consistent when measuring only one observable?

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Abstract

Based upon our assertion, there is an inconsistency in von Neumann's theory. Barros discusses the inconsistencies do not come from von Neumann's theory, but from extra assumptions about the reality of observables. von Neumann's theory is equivalent to Newton's theory when we consider only commuting observables. Using this fact, we discuss there is an inconsistency, probably due to the nature of Matrix theory based on non-commutativeness, within von Neumann's theory. That is, we may omit extra assumptions about the reality of observables. The main result is that von Neumann's theory is not consistent when measuring only one observable.

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I. INTRODUCTION

von Neumann's theory (cf. [1–5]) is a physical theory. Recently, Nagata and Nakamura claim [6, 7] to derive an inconsistency in von Neumann's theory. Barros discusses [8] that the inconsistencies do not come from von Neumann's theory, but from extra assumptions about the reality of observables. We discuss the inconsistency comes from von Neumann's theory, without extra assumptions about the reality of observables. We show here the inconsistency in an arbitrary dimensional unitary space when measuring commuting observables/an observable, which is based on Newton's theory.

We notice that von Neumann's mathematical model for quantum mechanics is quite logically successful. And the axiomatic system for the mathematical model is a very consistent one. Thus, we cannot say that von Neumann's mathematical model has an inconsistency. What is the inconsistency to be discussed in this paper? We cannot expand the von Neumann's beautiful mathematical model more in handling real experimental data. Mathematically, von Neumann's model is logically very consistent, which fact is true. However, von Neumann's theory is questionable in the sense that the mathematical model does not always expand to real experimental data. And there is the inconsistency if we apply the von Neumann's model to expanding even a simple physical situation. In short, von Neumann's mathematical model might not be useful in that case.

The inconsistency to be discussed in this paper is very impressive. von Neumann's mathematical model has the qualification to be very true axiomatic system for quantum mechanics. Therefore, we cannot modify the axioms based on the nature of Matrix theory. Nevertheless, we encounter an inconsistency, probably due to the nature of Matrix theory based on non-commutativeness, within von Neumann's theory.

Here, we discuss there is an inconsistency within von Neumann's theory even for commuting observables. We do not introduce extra assumptions about the reality of observables because we consider only commuting observables. We suppose the two measured observables are commutative. We introduce a supposition that the operation Addition is equivalent to the operation Multiplication and we have an example of an inconsistency, probably due to the nature of Matrix theory based on non-commutativeness. Finally, we discuss von Neumann's theory is not consistent when measuring only one observable.

II. VON NEUMANN'S THEORY FOR TWO COMMUTING OBSERVABLES

Though doing later, we dare to introduce firstly a supposition that the sum rule is equivalent to the product rule [9, 10] for the purpose of showing our interesting objective obtained here [11]. The supposition that the sum rule is equivalent to the product rule means a supposition that the operation Addition is equivalent to the operation Multiplication (see [12]).

Let A_1, A_2 be two Hermitian operators, where they are also supposed to be commutative. They could be defined respectively as follows:

$$A_1 \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad A_2 \equiv \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{1}$$

Let us consider a simultaneous eigenstate of A_1, A_2, that is, $|\Psi\rangle$, such that

$$\langle\Psi|A_1|\Psi\rangle = +1, \quad \langle\Psi|A_2|\Psi\rangle = -1. \tag{2}$$

Thus, the measured results of trials are either $+1$ or -1.

First, we define the functional rule as follows:

$$f(g(O)) = g(f(O)), \tag{3}$$

where O is an Hermitian operator and f, g are appropriate functions. Second, the sum rule is defined as follows:

$$f(A_1 + A_2) = f(A_1) + f(A_2). \tag{4}$$

Finally, the product rule is defined as follows:

$$f(A_1 \cdot A_2) = f(A_1) \cdot f(A_2). \tag{5}$$

This fact above is based on the property of these two Hermitian operators themselves. This leads to the propositions that they are valid even for the real numbers of the diagonal elements of the two Hermitian operators.

We may have [3, 13, 14] the following relation between the three rules which are valid for the commuting observables:

$$\begin{aligned} &\text{The functional rule} \\ &\Leftrightarrow \text{The sum rule} \\ &\Leftrightarrow \text{The product rule} \end{aligned} \tag{6}$$

The Kochen-Specker theorem says the situation that some quantum observables do not commute [15]. Our argumentations can be based on the two commuting observables. We may introduce a hidden variable theory in handling real experimental data [16]. We can introduce the supposition that the sum rule is equivalent to the product rule when all the measured observables commute simultaneously. Notice we consider only the two commuting observables here.

III. VON NEUMANN'S THEORY IS NOT CONSISTENT

We might be in an inconsistency when the first result is $+1$ by the measured observable A_1, the second result is -1 by the measured observable A_2, and then $[A_1, A_2] = 0$. In general, the physical situation is either $[A_1, A_2] \neq 0$ or $[A_1, A_2] = 0$. However we may be in the inconsistency when we suppose $[A_1, A_2] = 0$, probably due to the nature of Matrix theory based on non-commutativeness.

We consider a value V which is the sum of two data in an experiment. The measured results of trials are either $+1$ or -1. We suppose the number of -1 is equal to the number of $+1$. If the number of trials is two, then we have

$$V = (+1) + (-1) = 0. \tag{7}$$

We derive a general necessary condition of the product $V \times V$ of the value V. In this general case, we have

$$V \times V = 0. \tag{8}$$

This is a general necessary condition for either $[A_1, A_2] \neq 0$ or $[A_1, A_2] = 0$.

We can depict experimental data r_1, r_2 as follows: $r_1 = +1$ and $r_2 = -1$. Let us write V as follows:

$$V = r_1 + r_2. \tag{9}$$

In the following, we evaluate a value $(V \times V)$ and derive a specific necessary condition under the supposition that the two measured observables are commuting. That is, $[A_1, A_2] = 0$.

We introduce a supposition that the sum rule is equivalent to the product rule [9, 10]. The supposition that the sum rule is equivalent to the product rule means a supposition that the operation Addition is equivalent to the operation Multiplication (see [12]). Then, we have

$$\begin{aligned} &V \times V \\ &= (r_1 + r_2) \times (r_1 + r_2) \\ &= (r_1 \times r_1) + (r_1 \times r_2) + (r_2 \times r_1) + (r_2 \times r_2) \\ &= (r_1)^2 + (r_1 + r_2) + (r_2 + r_1) + (r_2)^2 \\ &= (r_1)^2 + (r_1 + r_1) + (r_2 + r_2) + (r_2)^2 \\ &= (r_1)^2 + (r_1 \times r_1) + (r_2 \times r_2) + (r_2)^2 \\ &= 2((r_1)^2 + (r_2)^2) \\ &= 2((+1)^2 + (-1)^2) = 4. \end{aligned} \tag{10}$$

Thus,

$$V \times V = 4. \tag{11}$$

This is possible for the specific case $[A_1, A_2] = 0$.

We cannot assign simultaneously the truth value "1" for the two suppositions (8) and (11) when $[A_1, A_2] = 0$. We derive the inconsistency when $[A_1, A_2] = 0$.

In summary, we have been in the inconsistency when the first result is $+1$, the second result is -1, and then $[A_1, A_2] = 0$.

IV. GENERAL CASE

Let us move ourselves into the more general case. Especially, we discuss von Neumann's theory is not consistent when measuring only one observable.

A. The first result is not equal to the second result

We might be in an inconsistency when the first result is x by the measured observable A_1, the second result is not x by the measured observable A_2, and then $[A_1, A_2] = 0$. In general, the physical situation is either $[A_1, A_2] \neq 0$ or $[A_1, A_2] = 0$. However we may be in the inconsistency when we suppose $[A_1, A_2] = 0$, probably due to the nature of Matrix theory based on non-commutativeness.

We consider a value V which is the sum of two data in an experiment. The measured results of trials are either x or $y (\neq x)$. We suppose the number of x is equal to the number of y. If the number of trials is two, then we have

$$V = x + y. \tag{12}$$

We derive a general necessary condition of the product $V \times V$ of the value V. In this general case, we have

$$V \times V = (x + y)^2. \tag{13}$$

This is a general necessary condition for either $[A_1, A_2] \neq 0$ or $[A_1, A_2] = 0$.

We can depict experimental data r_1, r_2 as follows: $r_1 = x$ and $r_2 = y$. Let us write V as follows:

$$V = r_1 + r_2. \tag{14}$$

In the following, we evaluate a value $(V \times V)$ and derive a specific necessary condition under the supposition that the two measured observables are commuting. That is, $[A_1, A_2] = 0$.

We introduce a supposition that the sum rule is equivalent to the product rule [9, 10]. The supposition that the sum rule is equivalent to the product rule means a supposition that the operation Addition is equivalent to the operation Multiplication (see [12]). Then, we have

$$\begin{aligned}
&V \times V \\
&= (r_1 + r_2) \times (r_1 + r_2) \\
&= (r_1 \times r_1) + (r_1 \times r_2) + (r_2 \times r_1) + (r_2 \times r_2) \\
&= (r_1)^2 + (r_1 + r_2) + (r_2 + r_1) + (r_2)^2 \\
&= (r_1)^2 + (r_1 + r_1) + (r_2 + r_2) + (r_2)^2 \\
&= (r_1)^2 + (r_1 \times r_1) + (r_2 \times r_2) + (r_2)^2 \\
&= 2((r_1)^2 + (r_2)^2) \\
&= 2(x^2 + y^2).
\end{aligned} \tag{15}$$

Thus,

$$V \times V = 2(x^2 + y^2). \tag{16}$$

This is possible for the specific case $[A_1, A_2] = 0$.

We cannot assign simultaneously the truth value "1" for the two suppositions (13) and (16) when $[A_1, A_2] = 0$. We derive the inconsistency when $[A_1, A_2] = 0$.

In summary, we have been in the inconsistency when the first result is x, the second result is not x, and then $[A_1, A_2] = 0$.

B. The first result is equal to the second result

We discuss von Neumann's theory is not consistent when measuring only one observable. We might be in an inconsistency when the first result is $x(\neq 0)$ by the measured observable A_1, the second result is also x by the measured observable A_2, and then $[A_1, A_2] = 0$. In general, the physical situation is either $[A_1, A_2] \neq 0$ or $[A_1, A_2] = 0$. However we may be in the inconsistency when we suppose $[A_1, A_2] = 0$, probably due to the nature of Matrix theory based on non-commutativeness. It may be that we measure only one observable A, ($A = A_1 = A_2$ and $x \neq 0$).

We have

$$V = x + x = 2x. \tag{17}$$

We derive a general necessary condition of the product $V \times V$ of the value V. In this general case, we have

$$V \times V = 4x^2. \tag{18}$$

This is a general necessary condition for either $[A_1, A_2] \neq 0$ or $[A_1, A_2] = 0$.

We can depict experimental data r_1, r_2 as follows: $r_1 = x$ and $r_2 = x$. Let us write V as follows:

$$V = r_1 + r_2. \tag{19}$$

In the following, we evaluate a value $(V \times V)$ and derive a specific necessary condition under the supposition that the two measured observables are commuting.

We introduce a supposition that the operation Addition is equivalent to the operation Multiplication. Then, we have

$$\begin{aligned}
&V \times V \\
&= (r_1 + r_2) \times (r_1 + r_2) \\
&= (r_1 \times r_1) + (r_1 \times r_2) + (r_2 \times r_1) + (r_2 \times r_2) \\
&= (r_1 + r_1) + (r_1 + r_2) + (r_2 + r_1) + (r_2 + r_2) \\
&= 8x.
\end{aligned} \tag{20}$$

Thus,

$$V \times V = 8x. \tag{21}$$

This is possible for the specific case $[A_1, A_2] = 0$.

When $x \neq 2$, we cannot assign simultaneously the truth value "1" for the two suppositions (18) and (21) when $[A_1, A_2] = 0$. We derive the inconsistency when $[A_1, A_2] = 0$.

Let us consider the case where $x = 2$. We introduce a supposition that the operation Addition is equivalent to the operation Multiplication. Then, we have

$$\begin{aligned}
&V \times V \\
&= (r_1 + r_2) \times (r_1 + r_2) \\
&= (r_1 \times r_1) + (r_1 \times r_2) + (r_2 \times r_1) + (r_2 \times r_2) \\
&= (r_1 \times r_1) + (r_1 + r_2) + (r_2 + r_1) + (r_2 \times r_2) \\
&= 2(x^2 + x).
\end{aligned} \tag{22}$$

Thus,

$$V \times V = 2(x^2 + x). \tag{23}$$

This is possible for the specific case $[A_1, A_2] = 0$.

When $x = 2$, we cannot assign simultaneously the truth value "1" for the two suppositions (18) and (23) when $[A_1, A_2] = 0$. We derive the inconsistency when $[A_1, A_2] = 0$.

Let us consider the case where $x = 0$. We see $0 + 0 = 0 \times 0 = 0$. Thus, the supposition that the operation Addition is equivalent to the operation Multiplication does not work in order to derive the inconsistency. Hence, we have always $V \times V = 0$ when $x = 0$. Thus, we cannot derive the inconsistency when $x = 0$.

In summary, we have been in the inconsistency when the first result is x, the second result is also x, and then $[A_1, A_2] = 0$. Especially, we have discussed von Neumann's theory is not consistent when measuring only one observable A, ($A = A_1 = A_2$ and $x \neq 0$).

V. CONCLUSIONS AND DISCUSSIONS

In conclusions, Nagata and Nakamura have claimed [6, 7] to derive an inconsistency in von Neumann's theory. Barros has discussed [8] as follows: The inconsistencies do not have come from von Neumann's theory, but from extra assumptions about the reality of observables. Here we have discussed there is an inconsistency, probably due to the nature of Matrix theory based on noncommutativeness, within von Neumann's theory even for commuting observables. We do not have introduced extra assumptions about the reality of observables because we consider only commuting observables. Finally, we have discussed von Neumann's theory is not consistent when measuring only one observable.

If the problem were simply an inconsistency, there are multiple logical systems that can cope with such a problem with robustness (see [17]).

Generally Multiplication is completed by Addition. Therefore, we think that Addition of the starting point may be superior to any other case.

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DECLARATIONS

Ethical Approval

We are in an applicable thought to Ethical Approval.

Competing interests

The authors state that there is no conflict of interest.

Authors' contributions

Koji Nagata and Tadao Nakamura wrote and read the manuscript.

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