Integer Sequences in the Collatz Graph

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Abstract

Let G be a graph with node V(G) represented by a sequence of odd integer constructed from the inverse iteration defined by $a_n = (2^{e(a_n)} a_{n+1} - 1)/3$, where $e(a_n) \in N$ is the highest exponent for which $2^{e(a_n)}$ exactly divide $3a_n + 1$. This graph is called the Collatz graph. The odd integer sequences inferred from the Collatz graph provide new insights into the validity of the Collatz conjecture.

1. **Introduction: the Collatz conjecture**

Denote by $N = \{1,2,3\ldots\ldots\ldots\}$, $N_0 = \{0,1,2,3,\ldots\ldots\ldots\}$, and $\mathbf{D}^+ = 2N_0 + 1$ the set of positive odd number. Let $[r]_q$ represents r(mod q), where r is least nonnegative residue modulo q. Define the recursive function introduced by Crandall [1] and Batang [2]:

$$a_{n+1} = (3a_n + 1)/2^{e(a_n)} \qquad (1)$$

where $e(a_n) \in N$ is the highest exponent for which $2^{e(a_n)}$ exactly divide $3a_n + 1$. . For an initial a_0, any k iteration on a_0 generate a sequence of odd integer , $\{a_0, a_1, \ldots\ldots\ldots a_k\}$. The collatz conjecture asserts that for every odd integer $a_0 > 1$, there exists $k \in N$ such that $a_k = 1$

2. **The Collatz graph**

The Collatz graph G= (V,E), with node V(G) and edge set E(G), based on the inverse iteration defined by

$$a_n = (2^{e(a_n)} a_{n+1} - 1)/3, \qquad (2)$$

where $e(a_n)$ is any positive integer such that $(2^{e(a_n)} a_{n+1} - 1) \equiv [0]_3$. One can show that $e(a_n) = 2n$ if $a_{n+1} \equiv [1]_3$ and $e(a_n) = 2n-1$ if $a_{n+1} \equiv [2]_3$ for $n \in N$, while $a_n = \emptyset$ if $a_{n+1} \equiv [0]_3$. Obviously a_{n+1} produces a sequence of odd integer starting with $(2^2 a_{n+1} - 1)/3$ or $(2^1 a_{n+1} - 1)/3$.

Starting from the trivial node $q_0 = \{1\}$, each sequence obtained by (2) corresponds to a node in G. The first node in the Collatz graph is obtained by $(2^{2n}(1)-1)/3$, $n \in N$. Thus, the first sequence or the first node is { 1, 5, 21,85,341,1365,………..}.

Obviously each element in the sequence is obtained by the iteration

$$s_{n+1} = 4s_n + 1 , \qquad (3)$$

where s_0 is the first element or a seed of the sequence ,e.g., $s_0 = 1$ is the seed of { 1, 5, 21,85,…}. The graph G is a directed graph in which there is only one directed path from any node to the trivial node $q_0 = \{1\}$ as shown in Figure 1. Each node is represented as $q_i \equiv \{ s_{n+1} = 4s_n + 1, n \in N_0 \mid s_0 = i, i \in D^+\}$, e.g., $q_1 = \{ 1, 5, 21, 85,..\}$. There are infinite levels in G. There is only one node in level 1 but there are infinite nodes in level 2,3,…∞. Let A be the union of all nodes in G. The aim is to show that

A- $\mathbf{D}^+$ is an empty set to validate the Collatz conjecture.

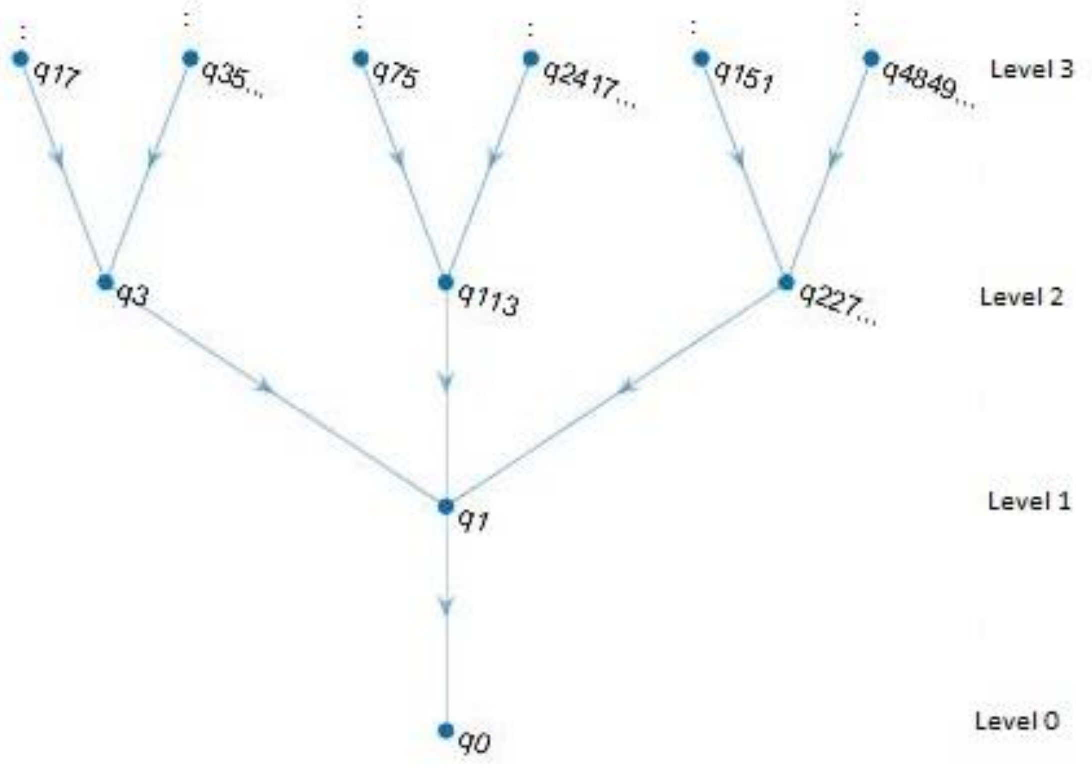

Figure 1. Stucture of G

3. Special subsets of $\mathbf{D^+}$

Let $q_0 = \{1\}$, obtain q_i, $i \in D^+$, $q_i \subset D^+$ as follows:

$q_i \equiv \{ s_{n+1} = 4s_n + 1,\ n \in N_0 \mid s_0 = i,\ i \in D^+.\ i \neq 5 + 8n,\ n \in N_0 \}$. The reason that any $s_0 \neq 5 + 8n$ is that odd integer $5 + 8n$ is already in some q_i, $i \neq 5 + 8n$, $n \in N_0$. q_i is represented as node in $\hat{G}$ and their relations among them are shown in Figure 2. Define B as the union of all q_i in $\hat{G}$, obviously B is equal to $\mathbf{D^+}$.

$q_0 \leftarrow q_1 \Leftarrow$ infinite nodes: q_3, q_{113}, ...

$q_1 \leftarrow q_3 \Leftarrow$ infinite nodes: q_{17}, q_{35},. ....

$q_{11} \leftarrow q_7 \Leftarrow$ infinite nodes :q_9, q_{19},. ....

$q_7 \leftarrow q_9 \Leftarrow$ infinite nodes: q_{49}, q_{99},. ....

$q_{17} \leftarrow q_{11} \Leftarrow$ infinite nodes: q_7, q_{241},. ....

$q_{23} \leftarrow q_{15} \Leftarrow$ infinite nodes: q_{81}, q_{163},. ....

$q_3 \leftarrow q_{17} \Leftarrow$ infinite nodes: q_{11}, q_{369},. ....

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Figure 2 Structure of $\hat{G}$

The validity of Collatz conjecture is established by proving that A-B is an empty set. This assertion will be proved by contradiction as follows:

Assume A-B = $\{ q_{n1}, q_{n2}, \ldots\ldots. q_{nm} \}$ is not an empty set. Since each q_{ni}, $i=1,2,\ldots m$, $m \in N$ connects to an infinite number of nodes as shown in Figure 2 then it is impossible to have a finite number of nodes among q_{ni} connected to each q_{ni}, $i=1,2,\ldots m$. Thus, A-B is an empty set.

References

[1] R . E. Crandall, " On the "3x+1" problem", Math. Of Comp. Vol. 32, N0. 144, October 1978, p. 1281-1292.

[2] Z. B. Batang, "Integer patterns in Collatz sequence", arXiv: 1907.07088v2 [math.GM] 17 Jul 2019.