A Novel Set of Representations for ζ(3)

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Abstract. Here, we provide a novel set of expressions for zeta (3) which are systematic results.

In our study, the following 19 expressions of zeta (3), called expressions of ***the first kind***, were obtained by an equation or system of equations. For reference, they are shown below.

Note: If one wishes to dig deeper, there are more than 520,000 expressions for zeta (3) called expressions of ***the second kind***.

№	Expressions of ***the First Kind*** (B_{2n} are Bernoulli numbers)
1	$\zeta(3) = \frac{4\cdot\pi^2}{35}\cdot\left[\frac{3}{2} + \ln\left(\frac{2}{\pi}\right) + \sum_{n=1}^{\infty}\frac{\left\lvert B_n\right\rvert\cdot(\pi)^{2n}}{2^{2n}\cdot n\cdot(2n+2)!}\right]$
2	$\zeta(3) = \frac{\pi^2}{7}\cdot\left[\frac{3}{2} - \ln(\pi) + \sum_{n=1}^{\infty}\frac{\left\lvert B_{2n}\right\rvert\cdot(\pi)^{2n}}{n\cdot(2n+2)!}\right]$
3	$\zeta(3) = \frac{36\cdot\pi^2}{35}\cdot\left[\frac{3}{2} + \ln\left(\frac{2}{3\cdot\pi}\right) + \sum_{n=1}^{\infty}\frac{3^{2n}\cdot\left\lvert B_n\right\rvert\cdot(\pi)^{2n}}{2^{2n}\cdot n\cdot(2n+2)!}\right]$
4	$\zeta(3) = \frac{\pi^2}{8}\cdot\left[\frac{77}{18} + \frac{7}{3}\cdot\ln\left(\frac{2}{\pi}\right) - \frac{9\cdot\ln(3)}{4} + \sum_{n=1}^{\infty}\frac{\left(1+3^{2n+3}\right)\cdot\left\lvert B_n\right\rvert\cdot(\pi)^{2n}}{2^{2n+7}\cdot n\cdot(2n+3)!}\right]$
5	$\zeta(3) = \frac{\pi^2}{2}\cdot\left[\frac{11}{18} - \frac{\ln(\pi)}{3} + \sum_{n=1}^{\infty}\frac{\left\lvert B_{2n}\right\rvert\cdot\pi^{2n}}{n\cdot(2n+3)!}\right]$
6	$\zeta(3) = \pi^2\cdot\left[\frac{11}{9} - \frac{2}{3}\cdot\ln(2\pi) + \sum_{n=1}^{\infty}\frac{2^{2n+1}\cdot\left\lvert B_{2n}\right\rvert\cdot\pi^{2n}}{n\cdot(2n+3)!}\right]$
7	$\zeta(3) = \frac{\pi^2}{9}\cdot\left[\frac{125}{48} - \frac{\ln(2)}{6} - \frac{5\cdot\ln(\pi)}{4} + \sum_{n=1}^{\infty}\frac{\left(17\cdot 2^{2n} - 2\right)\cdot\left\lvert B_{2n}\right\rvert\cdot(\pi)^{2n}}{2^{2n}\cdot n\cdot(2n+4)!}\right]$
8	$\zeta(3) = \frac{\pi^2}{55}\cdot\left[\frac{3625}{144} + \frac{27}{2}\cdot\ln\left(\frac{2}{3}\right) - \frac{145\ln(\pi)}{12} + \sum_{n=1}^{\infty}\frac{\left(2\cdot 3^{2n+4} - 17\cdot 2^{2n}\right)\cdot\left\lvert B_n\right\rvert\cdot(\pi)^{2n}}{2^{2n}\cdot n\cdot(2n+4)!}\right]$

9	$\zeta(3) = \frac{\pi^2}{8}\cdot\left[\frac{125}{36} + \frac{5}{3}\cdot\ln\left(\frac{2}{\pi}\right) - \frac{27\cdot\ln(3)}{16} + \sum_{n=1}^{\infty}\frac{\left(3^{2n+4} - 1\right)\cdot\left\|B_n\right\|\cdot(\pi)^{2n}}{2^{2n+2}\cdot n\cdot(2n+4)!}\right]$
10	$\zeta(3) = \pi^2\left[\frac{25}{36} - \frac{\ln(2\pi)}{3} + \sum_{n=1}^{\infty}\frac{2^{2n+2}\cdot\left\|B_{2n}\right\|\cdot\pi^{2n}}{n\cdot(2n+4)!}\right]$
11	$\zeta(3) = \frac{\pi^2}{16}\cdot\left[\frac{137}{20} + \frac{61\cdot\ln(2)}{15} - \frac{81\cdot\ln(3)}{20} - 3\cdot\ln(\pi) + \sum_{n=1}^{\infty}\frac{\left(3^{2n+5} + 1 - 2^{2n+6}\right)\cdot\left\|B_n\right\|\cdot(\pi)^{2n}}{2^{2n+4}\cdot n\cdot(2n+5)!}\right]$
12	$\zeta(3) = \frac{\pi^2}{16}\cdot\left[\frac{1781}{180} - \frac{317\cdot\ln(2)}{45} + \frac{27\cdot\ln(3)}{20} - \frac{13\cdot\ln(\pi)}{3} + \sum_{n=1}^{\infty}\frac{\left(2^{4n+10} + 1 - 3^{2n+5}\right)\cdot\left\|B_n\right\|\cdot(\pi)^{2n}}{3\cdot 2^{2n+4}\cdot n\cdot(2n+5)!}\right]$
13	$\zeta(3) = \pi^2\cdot\left[\frac{137}{240} - \frac{4\ln(2)}{15} - \frac{\ln(\pi)}{4} + \sum_{n=1}^{\infty}\frac{\left(2^{2n+4} - 1\right)\cdot\left\|B_{2n}\right\|\cdot(\pi)^{2n}}{n\cdot(2n+5)!}\right]$
14	$\zeta(3) = \frac{\pi^2}{272}\cdot\left[\frac{6517}{60} + \frac{607\cdot\ln(2)}{9} - \frac{2673\cdot\ln(3)}{40} - \frac{133\cdot\ln(\pi)}{3} + \sum_{n=1}^{\infty}\frac{\left(11\cdot 3^{2n+7} + 223 - 65\cdot 2^{2n+7}\right)\cdot\left\|B_n\right\|\cdot(\pi)^{2n}}{2^{2n}\cdot n\cdot(2n+6)!}\right]$
15	$\zeta(3) = \frac{\pi^2}{103}\cdot\left[\frac{1421}{24} - \frac{2513\cdot\ln(2)}{45} + \frac{81\cdot\ln(3)}{5} - \frac{145\cdot\ln(\pi)}{6} + \sum_{n=1}^{\infty}\frac{\left(223\cdot 2^{4\cdot n+6} + 65\cdot 2^{2\cdot n+2} - 2^3\cdot 3^{2n+6}\right)\cdot\left\|B_n\right\|\cdot(\pi)^{2n}}{2^{2n}\cdot n\cdot(2n+6)!}\right]$
16	$\zeta(3) = \frac{\pi^2}{16}\cdot\left[\frac{539}{60} - \frac{347\cdot\ln(2)}{45} + \frac{81\cdot\ln(3)}{40} - \frac{11\cdot\ln(\pi)}{3} + \sum_{n=1}^{\infty}\frac{\left[2^{(4n+11)} - 1 - 3^{2n+6}\right]\cdot\left\|B_n\right\|\cdot(\pi)^{2n}}{2^{2n}\cdot n\cdot(2n+6)!}\right]$
17	$\zeta(3) = \frac{\pi^2}{25}\cdot\left[\frac{1519}{120} - \frac{263\cdot\ln(2)}{45} - \frac{31\cdot\ln(\pi)}{6} + \sum_{n=1}^{\infty}\frac{\left(33\cdot 2^{4\cdot n+6} + 8 - 65\cdot 2^{2n+2}\right)\cdot\left\|B_n\right\|\cdot(\pi)^{2n}}{2^{2n}\cdot n\cdot(2n+6)!}\right]$
18	$\zeta(3) = \frac{\pi^2}{4}\cdot\left[\frac{121}{56} - \frac{1571\cdot\ln(2)}{630} + \frac{243\cdot\ln(3)}{280} - \frac{5\cdot\ln(\pi)}{6} + \sum_{n=1}^{\infty}\frac{\left(2^{4\cdot n+12} + 3\cdot 2^{2\cdot n+6} - 1 - 3^{2n+7}\right)\cdot\left\|B_n\right\|\cdot(\pi)^{2n}}{2^{2n}\cdot n\cdot(2n+7)!}\right]$
19	$\zeta(3) = \frac{\pi^2}{44}\cdot\left[\frac{5327}{240} - \frac{5909\cdot\ln(2)}{210} + \frac{23571\cdot\ln(3)}{2240} - \frac{49\cdot\ln(\pi)}{6} + \sum_{n=1}^{\infty}\frac{\left(65\cdot 2^{4\cdot n+14} + 257\cdot 2^{2\cdot n+8} - 415 - 97\cdot 3^{2n+8}\right)\cdot\left\|B_{2n}\right\|\cdot(\pi)^{2n}}{3\cdot 2^{2n}\cdot n\cdot(2n+8)!}\right]$

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