On Prime Numbers Between kn And $(k+1)n$

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Abstracts

In this paper along with three previous studies on analyzing the binomial coefficients, we will complete the proof of a theorem. The theorem states that for two positive integers $n \geq 1$ and $k \geq 1$, if $n \geq k - 1$, then there always exists at least a prime number p such that $kn < p \leq (k + 1)n$. The Bertrand-Chebyshev's theorem is a special case of this theorem when $k = 1$.

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1. Introduction

The Bertrand-Chebyshev's theorem States that for any positive integer n, there is always a prime number p such that $n < p \le 2n$. It was proved by Pafnuty Chebyshev in 1850 [1]. In 2006, M. El Bachraoui [2] expanded the theorem by proving that for any positive integer n, there is a prime number p such that $2n < p \le 3n$. In 2011, Andy Loo [3] expanded the theorem to prove that there is a prime number in the interval $(3n, 4n)$ when $n \ge 2$. It comes up with a question: Does any positive integer k make $kn < p \le (k+1)n$ stand? If it does, in what conditions? Previously, the author partially answered these questions by analyzing the binomial coefficients $\binom{3n}{n}$, $\binom{4n}{n}$, and $\binom{\lambda n}{n}$ where λ is a positive integer [4] [5] [6]. In this paper, we will complete the work with the above methodology. In this section, we will cite some important concepts from the previous papers. Then in section 2 and section 3, we will fill up the gaps of λ from 5 to 25. And in section 4, we will convert λ to k to complete this paper.

From [4]:

Definition: $\Gamma_{a\ge p>b}\{n\}$ denotes the prime number decomposition operator. It is the product of the prime numbers in the decomposition of a positive integer n or a positive integer expression. In this operator, p is a prime number, a and b are real numbers, and $n \ge a \ge p > b \ge 1$.
It has some properties:
It is always true that $\Gamma_{a\ge p\ge b}\{n\} \ge 1$. **— (1.1)**

If no prime number in $\Gamma_{a\ge p>b}\{n\}$, then $\Gamma_{a\ge p>b}\{n\} = 1$, or vice versa, if $\Gamma_{a\ge p>b}\{n\} = 1$, then no prime number in $\Gamma_{a\ge p>b}\{n\}$ as in $\Gamma_{12\ge p>4}\{12\} = 11^0 \cdot 7^0 \cdot 5^0 = 1$. **— (1.2)**

If there is at least one prime number in $\Gamma_{a\ge p>b}\{n\}$, then $\Gamma_{a\ge p>b}\{n\} > 1$, or vice versa, if $\Gamma_{a\ge p>b}\{n\} > 1$, then there is at least one prime number in $\Gamma_{a\ge p>b}\{n\}$ as in $\Gamma_{4\ge p>2}\{12\} = 3 > 1$. **— (1.3)**

For every positive integer n, there exists at least a prime number p such that $2n < p \le 3n$. **— (1.4)**

From [5]:

For every integer $n > 1$, there exists at least a prime number p such that $3n < p \le 4n$. **— (1.5)**

From [6 pp2-5]:

For $n \ge 2$ and $\lambda \ge 3$, $\binom{\lambda n}{n} > \dfrac{\lambda^{\lambda n-\lambda+1}}{n(\lambda-1)^{(\lambda-1)n-\lambda+1}}$ **— (1.6)**

We define $R(p)$ by the inequalities $p^{R(p)} \le \lambda n < p^{R(p)+1}$. If p divides $\binom{\lambda n}{n}$, then $v_p\left(\binom{\lambda n}{n}\right) \le R(p) \le \log_p(\lambda n)$, or $p^{v_p\left(\binom{\lambda n}{n}\right)} \le p^{R(p)} \le \lambda n$ **— (1.7)**

If $\lambda n \ge p > \lfloor\sqrt{\lambda n}\rfloor$, then $0 \le v_p\left(\binom{\lambda n}{n}\right) \le R(p) \le 1$ **— (1.8)**

For $n \geq (\lambda - 2) \geq 24$, there exists at least a prime number p such that $(\lambda - 1)n < p \leq \lambda n$. **— (1.9)**

Let $\pi(n)$ be the number of distinct prime numbers less than or equal to n. Among the first six consecutive natural numbers are three prime numbers 2, 3 and 5. Then, for each additional six consecutive natural numbers, at most one can add two prime numbers, $p \equiv 1$ (MOD 6) and $p \equiv 5$ (MOD 6).

Thus, $\pi(n) \leq \left\lfloor \frac{n}{3} \right\rfloor + 2 \leq \frac{n}{3} + 2$ **— (1.10)**

when $n > \lfloor\sqrt{\lambda n}\rfloor$, $\binom{\lambda n}{n} = \Gamma_{\lambda n \geq p > n}\{\frac{(\lambda n)!}{n! \cdot ((\lambda-1)n)!}\} \cdot \Gamma_{n \geq p > \lfloor\sqrt{\lambda n}\rfloor}\{\frac{(\lambda n)!}{n! \cdot ((\lambda-1)n)!}\} \cdot \Gamma_{\lfloor\sqrt{\lambda n}\rfloor \geq p}\{\frac{(\lambda n)!}{n! \cdot ((\lambda-1)n)!}\}$

when $n \leq \lfloor\sqrt{\lambda n}\rfloor$, $\binom{\lambda n}{n} \leq \Gamma_{\lambda n \geq p > n}\{\frac{(\lambda n)!}{n! \cdot ((\lambda-1)n)!}\} \cdot \Gamma_{\lfloor\sqrt{\lambda n}\rfloor \geq p}\{\frac{(\lambda n)!}{n! \cdot ((\lambda-1)n)!}\}$

Thus, $\binom{\lambda n}{n} \leq \Gamma_{\lambda n \geq p > n}\{\frac{(\lambda n)!}{n! \cdot ((\lambda-1)n)!}\} \cdot \Gamma_{n \geq p > \lfloor\sqrt{\lambda n}\rfloor}\{\frac{(\lambda n)!}{n! \cdot ((\lambda-1)n)!}\} \cdot \Gamma_{\lfloor\sqrt{\lambda n}\rfloor \geq p}\{\frac{(\lambda n)!}{n! \cdot ((\lambda-1)n)!}\}$ **— (1.11)**

$\Gamma_{\lambda n \geq p > n}\{\frac{(\lambda n)!}{n! \cdot ((\lambda-1)n)!}\} = \Gamma_{\lambda n \geq p > n}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\}$ since all prime numbers in $n!$ do not appear in the range of $\lambda n \geq p > n$.

Referring to **(1.8)**, $\Gamma_{n \geq p > \lfloor\sqrt{\lambda n}\rfloor}\{\frac{(\lambda n)!}{n! \cdot ((\lambda-1)n)!}\} < \prod_{n \geq p} p$. It has been proved [7] that for $n \geq 3$,

$\prod_{n \geq p} p < 2^{2n-3}$. Thus, for $n \geq 3$ and $\lambda \geq 3$, $\Gamma_{n \geq p > \lfloor\sqrt{\lambda n}\rfloor}\{\frac{(\lambda n)!}{n! \cdot ((\lambda-1)n)!}\} < \prod_{n \geq p} p < 2^{2n-3}$.

Referring to **(1.7)** and **(1.10)**, $\Gamma_{\lfloor\sqrt{\lambda n}\rfloor \geq p}\{\frac{(\lambda n)!}{n! \cdot ((\lambda-1)n)!}\} \leq (\lambda n)^{\frac{\sqrt{\lambda n}}{3} + 2}$

Thus for $n \geq 3$ and $\lambda \geq 3$, $\binom{\lambda n}{n} < \Gamma_{\lambda n \geq p > n}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\} \cdot 2^{2n-3} \cdot (\lambda n)^{\frac{\sqrt{\lambda n}}{3} + 2}$ **— (1.12)**

Applying **(1.6)** to **(1.12)**, when $n \geq 3$ and $\lambda \geq 3$, we have

$$\frac{\lambda^{\lambda n - \lambda + 1}}{n(\lambda-1)^{(\lambda-1)n-\lambda+1}} < \binom{\lambda n}{n} < \Gamma_{\lambda n \geq p > n}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\} \cdot 2^{2n-3} \cdot (\lambda n)^{\frac{\sqrt{\lambda n}}{3} + 2}.$$

Since $2^{2n-3} > 0$ and $(\lambda n)^{\frac{\sqrt{\lambda n}}{3} + 2} > 0$, when $n \geq 3$, and $\lambda \geq 3$,

$$\Gamma_{\lambda n \geq p > n}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\} > \frac{\lambda^{\lambda n - \lambda + 1}}{(\lambda n)^{\frac{\sqrt{\lambda n}}{3} + 2} \cdot 2^{2n-3} \cdot n(\lambda-1)^{(\lambda-1)n-\lambda+1}} = \frac{2\lambda^2 \cdot \left(\left(\frac{\lambda}{4}\right) \cdot \left(\frac{\lambda}{\lambda-1}\right)^{\lambda-1}\right)^{(n-1)}}{(\lambda n)^{\frac{\sqrt{\lambda n}}{3} + 3}}$$ **— (1.13)**

2. A Prime Number Between (λ-1)n and λn when 5 ≤ λ ≤ 7 and n ≥ λ-2

Proposition 1: For $n \geq 36$ and $5 \leq \lambda \leq 7$, there exists at least a prime number p such that $(\lambda - 1)n < p \leq \lambda n$. **— (2.1)**

Referring to **(1.13)**, when $n \geq 36$ and $5 \leq \lambda \leq 7$, we have

$$\Gamma_{\lambda n \geq p > n}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\} > \frac{2\lambda^2 \cdot \left(\left(\frac{\lambda}{4}\right) \cdot \left(\frac{\lambda}{\lambda-1}\right)^{\lambda-1}\right)^{(n-1)}}{(\lambda n)^{\frac{\sqrt{\lambda n}}{3}+3}} \qquad \text{— (2.2)}$$

Let $f_1(x) = \frac{2\lambda^2 \cdot \left(\left(\frac{\lambda}{4}\right) \cdot \left(\frac{\lambda}{\lambda-1}\right)^{\lambda-1}\right)^{(x-1)}}{(\lambda x)^{\frac{\sqrt{\lambda x}}{3}+3}}$ where x is a real number, the variable, and λ is a constant at one of the 3 integers from 5 to 7.

$f_1'(x) = f_1(x) \cdot \left(ln\left(\frac{\lambda}{4}\right) + ln\left(\frac{\lambda}{\lambda-1}\right)^{\lambda-1} - \frac{\sqrt{\lambda}\,(ln(x)+ln(\lambda)+2)}{6\sqrt{x}} - \frac{3}{x}\right) = f_1(x) \cdot f_2(x)$ where

$f_2(x) = ln\left(\frac{\lambda}{4}\right) + ln\left(\frac{\lambda}{\lambda-1}\right)^{\lambda-1} - \frac{\sqrt{\lambda}\,(ln(x)+ln(\lambda)+2)}{6\sqrt{x}} - \frac{3}{x}$

$f_2'(x) = \frac{\sqrt{\lambda}\,ln(\lambda)}{12x\sqrt{x}} + \frac{\sqrt{\lambda}\,ln(x)}{12x\sqrt{x}} + \frac{3}{x^2} > 0$ for $x > 1$ and $\lambda > 1$. Thus, $f_2(x)$ is a strictly increasing function.

When $x = 36$ and $\lambda = 5$, $f_2(x) = ln\left(\frac{5}{4}\right) + ln\left(\frac{5}{5-1}\right)^{5-1} - \frac{\sqrt{5}\,(ln(36)+ln(5)+2)}{6\sqrt{36}} - \frac{3}{36} \approx 0.5859 > 0$.

When $x = 36$ and $\lambda = 6$, $f_2(x) = ln\left(\frac{6}{4}\right) + ln\left(\frac{6}{6-1}\right)^{6-1} - \frac{\sqrt{6}\,(ln(36)+ln(6)+2)}{6\sqrt{36}} - \frac{3}{36} \approx 0.7155 > 0$.

When $x = 36$ and $\lambda = 7$, $f_2(x) = ln\left(\frac{7}{4}\right) + ln\left(\frac{7}{7-1}\right)^{7-1} - \frac{\sqrt{7}\,(ln(36)+ln(7)+2)}{6\sqrt{36}} - \frac{3}{36} \approx 08522 > 0$.

Since $f_2(x) > 0$ when $x = 36$ and $5 \leq \lambda \leq 7$, and since $f_2(x)$ is a strictly increasing function, then when $x \geq 36$ and $5 \leq \lambda \leq 7$, we have $f_2(x) > 0$. **— (2.3)**

Since when $x = 36$ and $5 \leq \lambda \leq 7$, $f_1(x) > 0$ and $f_2(x) > 0$, and $f_2(x)$ is a strictly increasing function, then $f_1'(x) = f_1(x) \cdot f_2(x) > 0$. Thus, when $x = 36$ and $5 \leq \lambda \leq 7$, $f_1(x)$ is a strictly increasing function. $f_1(x+1) > f_1(x)$. **— (2.4)**

When $\lambda = 5$ and $x = 36$, $f_1(x) = \frac{50 \cdot \left(\left(\frac{5}{4}\right) \cdot \left(\frac{5}{5-1}\right)^{5-1}\right)^{(36-1)}}{(180)^{\frac{\sqrt{180}}{3}+3}} = \frac{4.5522\text{E}+18}{7.1073\text{E}+16} > 1$.

When $\lambda = 6$ and $x = 36$, $f_1(x) = \frac{72 \cdot \left(\left(\frac{6}{4}\right) \cdot \left(\frac{6}{6-1}\right)^{6-1}\right)^{(36-1)}}{(216)^{\frac{\sqrt{216}}{3}+3}} = \frac{7.5378\text{E}+21}{2.7530\text{E}+18} > 1$.

When $\lambda = 7$ and $x = 36$, $f_1(x) = \frac{98 \cdot \left(\left(\frac{7}{4}\right) \cdot \left(\frac{7}{7-1}\right)^{7-1}\right)^{(36-1)}}{(252)^{\frac{\sqrt{252}}{3}+3}} = \frac{3.6007\text{E}+24}{8.1511\text{E}+19} > 1$.

Referring to **(2.4)**, when $x \geq 36$ and $5 \leq \lambda \leq 7$, $f_1(x) > 1$.

Let $x = n$, then when $n \geq 36$ and $5 \leq \lambda \leq 7$, $f_1(n) = \dfrac{2\lambda^2 \cdot \left(\left(\frac{\lambda}{4}\right) \cdot \left(\frac{\lambda}{\lambda-1}\right)^{\lambda-1}\right)^{(n-1)}}{(\lambda n)^{\frac{\sqrt{\lambda n}}{3}+3}} > 1$.

Thus, referring to **(2.2)**, when $n \geq 36$ and $5 \leq \lambda \leq 7$, $\Gamma_{\lambda n \geq p > n}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\} > 1$. **— (2.5)**

Referring to **(1.3)**, there exists at least a prime number p such that $n < p \leq \lambda n$.

Since $n > \lambda - 2$, in $\Gamma_{\lambda n \geq p > n}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\}$, $p \geq n + 1 = \sqrt{n^2 + 2n + 1} > \sqrt{(n+2)n} > \lfloor\sqrt{\lambda n}\rfloor$.

Referring to **(1.8)**, we have $0 \leq v_p\left(\Gamma_{\lambda n \geq p > n}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\}\right) \leq R(p) \leq 1$.

$\Gamma_{\lambda n \geq p > n}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\} =$

$= \Gamma_{\lambda n \geq p > (\lambda-1)n}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\} \cdot \prod_{i=1}^{i=\lambda-2}\left(\Gamma_{\frac{(\lambda-1)n}{i} \geq p > \frac{\lambda n}{i+1}}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\} \cdot \Gamma_{\frac{\lambda n}{i+1} \geq p > \frac{(\lambda-1)n}{i+1}}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\}\right)$

In $\prod_{i=1}^{\lambda-2}\left(\Gamma_{\frac{(\lambda-1)n}{i} \geq p > \frac{\lambda n}{i+1}}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\}\right)$, every distinct prime number p in this range in the numerator $(\lambda n)!$ has the form of $(i)! \cdot p^i$. It also has the same form of $(i)! \cdot p^i$ in the denominator $((\lambda-1)n)!$. They cancel to each other. Thus, referring to **(1.2)**,

$\prod_{i=1}^{\lambda-2}\left(\Gamma_{\frac{(\lambda-1)n}{i} \geq p > \frac{\lambda n}{i+1}}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\}\right) = 1$. Therefore, when $n \geq 36$ and $5 \leq \lambda \leq 7$,

$\Gamma_{\lambda n \geq p > n}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\} = \Gamma_{\lambda n \geq p > (\lambda-1)n}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\} \cdot \prod_{i=1}^{i=\lambda-2}\left(\Gamma_{\frac{\lambda n}{i+1} \geq p > \frac{(\lambda-1)n}{i+1}}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\}\right) > 1$.

— (2.6)

From **(1.1)**, $\Gamma_{\lambda n \geq p > (\lambda-1)n}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\} \geq 1$ and $\prod_{i=1}^{i=\lambda-2}\left(\Gamma_{\frac{\lambda n}{i+1} \geq p > \frac{(\lambda-1)n}{i+1}}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\}\right) \geq 1$, and in **(2.6)** at last one of these two parts is greater than 1.

When $n \geq 36$ and $5 \leq \lambda \leq 7$, if $\Gamma_{\lambda n \geq p > (\lambda-1)n}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\} > 1$, then referring to **(1.3)**, there exists at least a prime number p such that $(\lambda-1)n < p \leq \lambda n$. **— (2.7)**

If $\prod_{i=1}^{i=\lambda-2}\left(\Gamma_{\frac{\lambda n}{i+1} \geq p > \frac{(\lambda-1)n}{i+1}}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\}\right) = 1$, then $\Gamma_{\lambda n \geq p > (\lambda-1)n}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\} > 1$. **— (2.8)**

If $\prod_{i=1}^{i=\lambda-2}\left(\Gamma_{\frac{\lambda n}{i+1} \geq p > \frac{(\lambda-1)n}{i+1}}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\}\right) > 1$, then at least one factor $\Gamma_{\frac{\lambda n}{i+1} \geq p > \frac{(\lambda-1)n}{i+1}}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\} > 1$.

When the factor $\Gamma_{\frac{\lambda n}{i+1} \geq p > \frac{(\lambda-1)n}{i+1}}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\} > 1$, let $y_{i+1} = \frac{n}{i+1}$, then $y_{i+1} \geq \frac{36}{i+1}$. We have

$\Gamma_{\lambda y_{i+1} \geq p > (\lambda-1)y_{i+1}}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\} > 1$. Thus, when $y_{i+1} \geq \frac{36}{i+1}$, there exists at least a prime number p such that $(\lambda-1) \cdot y_{i+1} < p \leq \lambda \cdot y_{i+1}$

Since $n > y_{i+1} \geq \frac{36}{i+1}$, there exists at least a prime number p such that $(\lambda - 1)n < p \leq \lambda n$.

Thus, If $\prod_{i=1}^{i=\lambda-2}\left(\Gamma_{\frac{\lambda n}{i+1} \geq p > \frac{(\lambda-1)n}{i+1}}\left\{\frac{(\lambda n)!}{((\lambda-1)n)!}\right\}\right) > 1$, then $\Gamma_{\lambda n \geq p > (\lambda-1)n}\left\{\frac{(\lambda n)!}{((\lambda-1)n)!}\right\} > 1$. **— (2.9)**

Referring to **(2.8)** and **(2.9)**, when $n \geq 36$ and $5 \leq \lambda \leq 7$, if $\prod_{i=1}^{i=\lambda-2}\left(\Gamma_{\frac{\lambda n}{i+1} \geq p > \frac{(\lambda-1)n}{i+1}}\left\{\frac{(\lambda n)!}{((\lambda-1)n)!}\right\}\right) \geq 1$, then $\Gamma_{\lambda n \geq p > (\lambda-1)n}\left\{\frac{(\lambda n)!}{((\lambda-1)n)!}\right\} > 1$. Thus, referring to **(1.3)**, there exists at least a prime number p such that $(\lambda - 1)n < p \leq \lambda n$. **— (2.10)**

In conclusion from **(2.5)**, **(2.7)**, **(2.10)**, when $n \geq 36$ and $5 \leq \lambda \leq 7$, then $\Gamma_{\lambda n \geq p > n}\left\{\frac{(\lambda n)!}{((\lambda-1)n)!}\right\} > 1$.

When $\Gamma_{\lambda n \geq p > n}\left\{\frac{(\lambda n)!}{((\lambda-1)n)!}\right\} > 1$, then $\Gamma_{\lambda n \geq p > (\lambda-1)n}\left\{\frac{(\lambda n)!}{((\lambda-1)n)!}\right\} > 1$, and there exists at least a prime number p such that $(\lambda - 1)n < p \leq \lambda n$. Thus, **Proposition 1** is proven.

Proposition 2: For $35 \geq n \geq \lambda - 2$ and $5 \leq \lambda \leq 7$, there exists at least a prime number p such that $(\lambda - 1)n < p \leq \lambda n$. **— (2.11)**

We use tables to prove **(2.11)**. **Table 1**, **Table 2**, and **Table 3** show that when λ = 5, 6, and 7, **Proposition 2** is correct. Thus, **(2.11)** is valid.

Table 1. When $\lambda = 5$ and $3 \leq n \leq 35$, a prime number exists in the range of $4n < p \leq 5n$

n	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
p	13	17	23	29	31	37	41	43	47	53	59	61	67	71	73	79	83
n	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	
p	89	97	101	103	107	109	113	127	131	137	139	149	151	157	163	167	

Table 2. When $\lambda = 6$ and $4 \leq n \leq 35$, a prime number exists in the range of $5n < p \leq 6n$

n	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
p	23	29	31	37	41	47	53	59	61	67	71	79	83	89	97	101
n	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35
p	103	107	113	127	131	137	139	149	151	157	163	167	173	179	181	191

Table 3. When $\lambda = 7$ and $5 \leq n \leq 35$, a prime number exists in the range of $6n < p \leq 7n$

n	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
p	31	37	43	53	59	61	67	73	79	89	97	101	103	109	127	131
n	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	
p	137	149	151	157	163	167	173	179	181	191	193	197	211	223	227	

Combining **(2.1)** and **(2.11)**, we have proven that when $5 \leq \lambda \leq 7$ and $n \geq \lambda - 2$, there exists at least a prime number p such that $(\lambda - 1)n < p \leq \lambda n$. **— (2.12)**

3. A Prime Number Between (λ-1)n and λn when 8 ≤ λ ≤ 25 and n ≥ λ-2

Proposition 3: For $n \geq 24$ and $8 \leq \lambda \leq 25$, there exists at least a prime number p such that $(\lambda - 1)n < p \leq \lambda n$. **— (3.1)**

Referring to **(1.13)**, when $n \geq 24$ and $8 \leq \lambda \leq 25$, we have

$$\Gamma_{\lambda n \geq p > n}\left\{\frac{(\lambda n)!}{((\lambda-1)n)!}\right\} > \frac{2\lambda^2 \cdot \left(\left(\frac{\lambda}{4}\right) \cdot \left(\frac{\lambda}{\lambda-1}\right)^{\lambda-1}\right)^{(n-1)}}{(\lambda n)^{\frac{\sqrt{\lambda n}}{3}+3}} \quad \text{— (3.2)}$$

Let $f_3(x) = \dfrac{2\lambda^2 \cdot \left(\left(\frac{\lambda}{4}\right) \cdot \left(\frac{\lambda}{\lambda-1}\right)^{\lambda-1}\right)^{(x-1)}}{(\lambda x)^{\frac{\sqrt{\lambda x}}{3}+3}}$ where x is a real number, the variable, and λ is a constant at one of the 18 integers from 8 to 25.

$f_3{}'(x) = f_3(x) \cdot \left(ln\left(\frac{\lambda}{4}\right) + ln\left(\frac{\lambda}{\lambda-1}\right)^{\lambda-1} - \frac{\sqrt{\lambda}\,(ln(x)+ln(\lambda)+2)}{6\sqrt{x}} - \frac{3}{x}\right) = f_3(x) \cdot f_4(x)$ where

$f_4(x) = ln\left(\frac{\lambda}{4}\right) + ln\left(\frac{\lambda}{\lambda-1}\right)^{\lambda-1} - \frac{\sqrt{\lambda}\,(ln(x)+ln(\lambda)+2)}{6\sqrt{x}} - \frac{3}{x}$

$f_4{}'(x) = \frac{\sqrt{\lambda}\,ln(\lambda)}{12x\sqrt{x}} + \frac{\sqrt{\lambda}}{12x\sqrt{x}} + \frac{3}{x^2} > 0$ for $x > 1$ and $\lambda > 1$. Thus, $f_4(x)$ is a strictly increasing function.

We now calculate the $f_4(x)$ values and list them in **Table 4** for $x = 24$ and $\lambda = 8, 9, 10, \ldots 25$.

Table 4. When $x = 24$ and λ from 8 to 25, $f_4(x) > 0$

λ	8	9	10	11	12	13	14	15	16
$f_4(x)$	0.805	0.876	0.935	0.985	1.028	1.064	1.096	1.124	1.148
λ	17	18	19	20	21	22	23	24	25
$f_4(x)$	1.168	1.186	1.202	1.215	1.227	1.237	1.246	1.253	1.259

Table 4 shows that when $x = 24$ and λ from 8 to 25, $f_4(x) > 0$. Since $f_3(x) > 0$ and $f_4(x) > 0$, and $f_4(x)$ is a strictly increasing function, when $x \geq 24$ and $8 \leq \lambda \leq 25$, $f_3{}'(x) = f_3(x) \cdot f_4(x) > 0$. Thus, under these conditions, $f_3(x)$ is a strictly increasing function, and $f_3(x+1) > f_3(x)$. **— (3.3)**

We now calculate the $f_3(x)$ values and list them in **Table 5** for $x = 24$ and $\lambda = 8, 9, 10, \ldots 25$.

Table 5. When $x = 24$ and λ from 8 to 25, $f_3(x) > 1$

λ	8	9	10	11	12	13	14	15	16
$f_5(x)$	9.366	19.132	31.150	42.517	50.475	53.571	51.866	46.527	39.386
λ	17	18	19	20	21	22	23	24	25
$f_5(x)$	31.212	23.760	17.383	12.287	8.421	5.633	3.679	2.536	1.481

Table 5 shows when $x = 24$ and λ from 8 to 25, $f_3(x) > 1$. Since $f_3(x+1) > f_3(x)$, when $x \geq 24$ and $8 \leq \lambda \leq 25$, $f_3(x) > 1$. **— (3.4)**

Let $x = n$, then when $n \geq 24$ and $8 \leq \lambda \leq 25$, $f_3(n) = \frac{2\lambda^2 \cdot \left(\left(\frac{\lambda}{4}\right)\cdot\left(\frac{\lambda}{\lambda-1}\right)^{\lambda-1}\right)^{(n-1)}}{(\lambda n)^{\frac{\sqrt{\lambda n}}{3}+3}} > 1$.

Thus, referring to **(3.2)**, when $n \geq 24$ and $8 \leq \lambda \leq 25$, $\Gamma_{\lambda n \geq p > n}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\} > 1$. **— (3.5)**

Referring to **(1.3)**, there exists at least a prime number p such that $n < p \leq \lambda n$.

Since $n > \lambda - 2$, in $\Gamma_{\lambda n \geq p > n}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\}$, $p \geq n+1 = \sqrt{n^2+2n+1} > \sqrt{(n+2)n} > \lfloor\sqrt{\lambda n}\rfloor$.

Referring to **(1.8)**, we have $0 \leq v_p\left(\Gamma_{\lambda n \geq p > n}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\}\right) \leq R(p) \leq 1$.

$\Gamma_{\lambda n \geq p > n}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\} =$

$= \Gamma_{\lambda n \geq p > (\lambda-1)n}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\} \cdot \prod_{i=1}^{i=\lambda-2}\left(\Gamma_{\frac{(\lambda-1)n}{i} \geq p > \frac{\lambda n}{i+1}}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\} \cdot \Gamma_{\frac{\lambda n}{i+1} \geq p > \frac{(\lambda-1)n}{i+1}}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\}\right)$

In $\prod_{i=1}^{\lambda-2}\left(\Gamma_{\frac{(\lambda-1)n}{i} \geq p > \frac{\lambda n}{i+1}}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\}\right)$, every distinct prime number p in this range in the numerator $(\lambda n)!$ has the form of $(i)! \cdot p^i$. It also has the same form of $(i)! \cdot p^i$ in the denominator $((\lambda-1)n)!$. They cancel to each other. Thus, referring to **(1.2)**, $\prod_{i=1}^{\lambda-2}\left(\Gamma_{\frac{(\lambda-1)n}{i} \geq p > \frac{\lambda n}{i+1}}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\}\right) = 1$. Therefore, when $n \geq 24$ and $8 \leq \lambda \leq 25$,

$\Gamma_{\lambda n \geq p > n}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\} = \Gamma_{\lambda n \geq p > (\lambda-1)n}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\} \cdot \prod_{i=1}^{i=\lambda-2}\left(\Gamma_{\frac{\lambda n}{i+1} \geq p > \frac{(\lambda-1)n}{i+1}}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\}\right) > 1$. **— (3.6)**

From **(1.1)**, $\Gamma_{\lambda n \geq p > (\lambda-1)n}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\} \geq 1$ and $\prod_{i=1}^{i=\lambda-2}\left(\Gamma_{\frac{\lambda n}{i+1} \geq p > \frac{(\lambda-1)n}{i+1}}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\}\right) \geq 1$, and in **(3.6)** at last one of these two parts is greater than 1.

When $n \geq 24$ and $8 \leq \lambda \leq 25$, if $\Gamma_{\lambda n \geq p > (\lambda-1)n}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\} > 1$, then referring to **(1.3)**, there exists at least a prime number p such that $(\lambda-1)n < p \leq \lambda n$. **— (3.7)**

If $\prod_{i=1}^{i=\lambda-2}\left(\Gamma_{\frac{\lambda n}{i+1} \geq p > \frac{(\lambda-1)n}{i+1}}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\}\right) = 1$, then $\Gamma_{\lambda n \geq p > (\lambda-1)n}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\} > 1$. **— (3.8)**

If $\prod_{i=1}^{i=\lambda-2}\left(\Gamma_{\frac{\lambda n}{i+1} \geq p > \frac{(\lambda-1)n}{i+1}}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\}\right) > 1$, then at least one factor $\Gamma_{\frac{\lambda n}{i+1} \geq p > \frac{(\lambda-1)n}{i+1}}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\} > 1$.

When the factor $\Gamma_{\frac{\lambda n}{i+1} \geq p > \frac{(\lambda-1)n}{i+1}}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\} > 1$, let $y_{i+1} = \frac{n}{i+1}$, then $y_{i+1} \geq \frac{24}{i+1}$. We have

$\Gamma_{\lambda y_{i+1} \geq p > (\lambda-1)y_{i+1}}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\} > 1$. Thus, when $y_{i+1} \geq \frac{24}{i+1}$, there exists at least a prime number p such that $(\lambda-1) \cdot y_{i+1} < p \leq \lambda \cdot y_{i+1}$

Since $n > y_{i+1} \geq \frac{24}{i+1}$, there exists at least a prime number p such that $(\lambda-1)n < p \leq \lambda n$.

Thus, If $\prod_{i=1}^{i=\lambda-2}\left(\Gamma_{\frac{\lambda n}{i+1}\geq p>\frac{(\lambda-1)n}{i+1}}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\}\right) > 1$, then $\Gamma_{\lambda n\geq p>(\lambda-1)n}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\} > 1$. **— (3.9)**

Referring to **(3.8)** and **(3.9)**, when $n \geq 24$ and $8 \leq \lambda \leq 25$, if $\prod_{i=1}^{i=\lambda-2}\left(\Gamma_{\frac{\lambda n}{i+1}\geq p>\frac{(\lambda-1)n}{i+1}}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\}\right) \geq 1$, then $\Gamma_{\lambda n\geq p>(\lambda-1)n}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\} > 1$. Thus, referring to **(1.3)**, there exists at least a prime number p such that $(\lambda-1)n < p \leq \lambda n$. **— (3.10)**

In conclusion from **(3.5)**, **(3.7)**, **(3.10)**, when $n \geq 24$ and $8 \leq \lambda \leq 25$, then $\Gamma_{\lambda n\geq p>n}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\} > 1$. When $\Gamma_{\lambda n\geq p>n}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\} > 1$, then $\Gamma_{\lambda n\geq p>(\lambda-1)n}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\} > 1$, and there exists at least a prime number p such that $(\lambda-1)n < p \leq \lambda n$. Thus, **Proposition 3** is proven.

Proposition 4: For $23 \geq n \geq \lambda - 2$ and $8 \leq \lambda \leq 25$, there exists at least a prime number p such that $(\lambda-1)n < p \leq \lambda n$. **— (3.11)**

We use tables to prove **(3.11)**. **Table 6**, **Table 7**, and **Table 8** show that when $8 \leq \lambda \leq 25$, **Proposition 4** is correct. Thus, **(3.11)** is valid.

Table 6. When $8 \leq \lambda \leq 11$ and $\lambda - 2 \leq n \leq 23$, a prime number between $(\lambda-1)n$ and λn

	$\boldsymbol{n}$	6	7	8	9	10	11	12	13	14	
	$\boldsymbol{7n}$	42	49	56	63	70	77	84	91	98	
	$\boldsymbol{p}$	**47**	**53**	**59**	**67**	**73**	**83**	**89**	**97**	**101**	$\boldsymbol{\lambda=8}$
	$\boldsymbol{8n}$	48	56	64	72	80	88	96	104	112	
$\boldsymbol{\lambda=9}$	$\boldsymbol{p}$		**61**	**71**	**79**	**83**	**97**	**101**	**107**	**113**	
	$\boldsymbol{9n}$		63	72	81	90	99	108	117	126	
	$\boldsymbol{p}$			**73**	**83**	**97**	**101**	**109**	**127**	**131**	$\boldsymbol{\lambda=10}$
	$\boldsymbol{10n}$			80	90	100	110	120	130	140	
$\boldsymbol{\lambda=11}$	$\boldsymbol{p}$				**97**	**103**	**113**	**127**	**139**	**151**	
	$\boldsymbol{11n}$				99	110	121	132	143	154	
	$\boldsymbol{n}$	15	16	17	18	19	20	21	22	23	
	$\boldsymbol{7n}$	105	112	119	126	133	140	147	154	161	
	$\boldsymbol{p}$	**107**	**113**	**127**	**131**	**137**	**149**	**151**	**157**	**163**	$\boldsymbol{\lambda=8}$
	$\boldsymbol{8n}$	120	128	136	144	152	160	168	176	184	
$\boldsymbol{\lambda=9}$	$\boldsymbol{p}$	**127**	**131**	**139**	**149**	**157**	**167**	**173**	**179**	**191**	
	$\boldsymbol{9n}$	135	144	153	162	171	180	189	198	207	
	$\boldsymbol{p}$	**137**	**151**	**163**	**167**	**181**	**191**	**193**	**199**	**211**	$\boldsymbol{\lambda=10}$
	$\boldsymbol{10n}$	150	160	170	180	190	200	210	220	230	
$\boldsymbol{\lambda=11}$	$\boldsymbol{p}$	**157**	**167**	**179**	**191**	**197**	**211**	**223**	**227**	**233**	
	$\boldsymbol{11n}$	165	176	187	198	209	220	231	242	253	

Table 7. When $12 \leq \lambda \leq 15$ and $\lambda - 2 \leq n \leq 23$, a prime number between $(\lambda - 1)n$ and λn

	$\boldsymbol{n}$	10	11	12	13	14	15	16	17	18	19	20	21	22	23
	$\boldsymbol{11n}$	110	121	132	143	154	165	176	187	198	209	220	231	242	253
$\boldsymbol{\lambda} = 12$	$\boldsymbol{p}$	**113**	**127**	**137**	**149**	**157**	**167**	**181**	**193**	**199**	**223**	**229**	**239**	**257**	**269**
	$\boldsymbol{12n}$	120	132	144	156	168	180	192	204	216	228	240	252	264	276
$\boldsymbol{\lambda} = 13$	$\boldsymbol{p}$		**139**	**151**	**163**	**173**	**183**	**197**	**211**	**227**	**233**	**241**	**263**	**271**	**281**
	$\boldsymbol{13n}$		143	156	169	182	195	208	221	234	247	260	273	286	299
$\boldsymbol{\lambda} = 14$	$\boldsymbol{p}$			**167**	**179**	**191**	**199**	**223**	**223**	**239**	**257**	**269**	**277**	**293**	**307**
	$\boldsymbol{14n}$			168	182	196	210	224	238	252	266	280	294	308	322
$\boldsymbol{\lambda} = 15$	$\boldsymbol{p}$				**191**	**199**	**211**	**229**	**239**	**263**	**271**	**283**	**307**	**311**	**331**
	$\boldsymbol{15n}$				195	210	225	240	255	270	285	300	315	330	345

Table 8. When $16 \leq \lambda \leq 25$ and $\lambda - 2 \leq n \leq 23$, a prime number between $(\lambda - 1)n$ and λn

	$\boldsymbol{n}$	14	15	16	17	18	19	20	21	22	23	
	$\boldsymbol{15n}$	210	225	240	255	270	285	300	315	330	345	
$\boldsymbol{\lambda} = 16$	$\boldsymbol{p}$	**223**	**227**	**241**	**257**	**277**	**293**	**313**	**317**	**331**	**347**	
	$\boldsymbol{16n}$	224	240	256	272	288	304	320	336	352	368	
	$\boldsymbol{p}$		**251**	**263**	**281**	**293**	**307**	**337**	**349**	**353**	**373**	$\boldsymbol{\lambda} = 17$
	$\boldsymbol{17n}$		255	272	289	306	323	340	357	374	391	
$\boldsymbol{\lambda} = 18$	$\boldsymbol{p}$			**277**	**293**	**311**	**331**	**347**	**359**	**379**	**397**	
	$\boldsymbol{18n}$			288	306	324	342	360	378	396	414	
	$\boldsymbol{p}$				**307**	**337**	**349**	**373**	**383**	**397**	**419**	$\boldsymbol{\lambda} = 19$
	$\boldsymbol{19n}$				323	342	361	380	399	418	437	
$\boldsymbol{\lambda} = 20$	$\boldsymbol{p}$					**347**	**367**	**389**	**401**	**421**	**439**	
	$\boldsymbol{20n}$					360	380	400	420	440	460	
	$\boldsymbol{p}$						**283**	**409**	**431**	**443**	**461**	$\boldsymbol{\lambda} = 21$
	$\boldsymbol{21n}$						399	420	441	462	483	
$\boldsymbol{\lambda} = 22$	$\boldsymbol{p}$							**433**	**449**	**463**	**487**	
	$\boldsymbol{22n}$							440	462	484	506	
	$\boldsymbol{p}$								**467**	**491**	**509**	$\boldsymbol{\lambda} = 23$
	$\boldsymbol{23n}$								483	506	529	
$\boldsymbol{\lambda} = 24$	$\boldsymbol{p}$									**521**	**541**	
	$\boldsymbol{24n}$									528	552	
	$\boldsymbol{p}$										**563**	$\boldsymbol{\lambda} = 25$
	$\boldsymbol{25n}$										575	

Combining **(3.1)** and **(3.11)**, we have proven that when $8 \leq \lambda \leq 25$ and $n \geq \lambda - 2$, there exists at least a prime number p such that $(\lambda - 1)n < p \leq \lambda n$. **— (3.12)**

4. A Prime Number Between kn and $(k+1)n$

From **(1.4),** for every positive integer n, there exists at least a prime number p such that $2n < p \leq 3n$.

From **(1.5),** for every integer $n > 1$, there exists at least a prime number p such that $3n < p \leq 4n$.

From **(2.12)**, when $5 \leq \lambda \leq 7$ and $n \geq \lambda - 2$, there exists at least a prime number p such that $(\lambda - 1)n < p \leq \lambda n$.

From **(3.12)**, when $8 \leq \lambda \leq 25$ and $n \geq \lambda - 2$, there exists at least a prime number p such that $(\lambda - 1)n < p \leq \lambda n$.

From **(1.9),** for $n \geq (\lambda - 2) \geq 24$, there exists at least a prime number p such that $(\lambda - 1)n < p \leq \lambda n$.

Combining **(1.4)**, **(1.5)**, **(2.12)**, **(3.12)**, and **(1.9)**, we show that for $n \geq \lambda - 2 \geq 1$, there exists at least a prime number p such that $(\lambda - 1)n < p \leq \lambda n$. **— (4.1)**

Let $k = \lambda - 1$, **(4.1)** becomes that for $n \geq k - 1 \geq 1$, there exists at least a prime number p such that $kn < p \leq (k + 1)n$. **— (4.2)**

Since the Bertrand-Chebyshev's theorem states that for any positive integer n, there is always a prime number p such that $n < p \leq 2n$, we can derive the **Theorem (4.3)**: For two positive integers $n \geq 1$ and $k \geq 1$, if $n \geq k - 1$, then there always exists at least a prime number p such that $kn < p \leq (k + 1)n$. The Bertrand-Chebyshev's theorem is a special case of **Theorem (4.3)** when $k = 1$.

5. References

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