PRIME FRIENDS AND THE SUMS OF PRIME FACTORS

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Abstract

When the sum of the prime factors of a composite number n results in another composite number m, we can take the sum of the prime factors of m. We can see if that is equal to yet another composite number s, and so on. Given any composite number n being greater than four, do we end up with every prime number via the method explained before? We present a constructive proof of this problem using number theory and logic. We find out that every prime number greater than three has finitely many prime friends (composite numbers whose prime factors added are the prime number itself). This is demonstrated through the use of the function $2k + 3$ and the partitions of prime numbers. In this way we also learn that we always end up with prime numbers through the sums of prime factors of composite numbers.

1 Introduction.

In this paper, we show that given any composite number $n > 4$, we will cover every prime number $p > 3$ by the theorem that involves summing up prime factors, which will follow.
The topic is quite similar to the concept of Ruth-Aaron numbers, where the sum of the prime factors of two individual numbers are equal to each other. These two numbers are what is known as a Ruth-Aaron pair; an example would be 714 and 715. The sum of the prime factors of 714 is indeed equal to the sum of the prime factors of 715, thus making them a pair. Though we will be working with prime numbers and composite numbers as a whole.

Theorem 1.

$$n = \prod_{i=1}^{k} p_i$$

$$\Rightarrow m = \sum_{i=1}^{k} p_i$$

If m is composite, then:
$m = p_a \cdot p_b \cdot p_c \cdot ... \cdot p_z$

$\Rightarrow s = p_a + p_b + p_c + ... + p_z$
If s is composite, then:
$s = p_\alpha \cdot p_\beta \cdot p_\gamma \cdot ... \cdot p_\omega$
.
.
.
$t = p_I \cdot p_{II} \cdot p_{III} \cdot ... \cdot p_r$
$\Rightarrow p = p_I + p_{II} + p_{III} + ... + p_r,$

where n is the composite number we started with and where $m, s, ...t$ are the other composite numbers for each step to the prime p.

We can see that $p \neq 2, 3$ due to their partitions:

$2 = 2 = 1 + 1$
$3 = 3 = 2 + 1 = 1 + 1 + 1,$

so there is no composite number t
where the sum of its prime factors is equal to 2 or 3.

4 is the only composite number the theorem does not work on. It is in a never-ending cyle with itself:

$4 = 2 \cdot 2$
$\Rightarrow 2 + 2 = 4$
.
.
.

Before proving the theorem completely, we prove that the sum of prime factors of n is always less than n. Even without proof, this seems trivial, though we have to be rigorous and careful in handling the complete proof.

Lemma 1.

The sum of the prime factors of a composite number n is always less than n if n is greater than four.

Proof:

Let $sum(n)$ be the sum of the prime factors of the composite number n.
Set p to be the smallest prime factor of n. It shows:

$sum(n) < n$, such that it can be proven that $n/p \geq p$ and that $p + sum(n/p) < n$.
So it can be shown:

$p + sum(n/p) < p + n/p \leq 2(n/p) \leq p(n/p) = n.$ $\square$

2 Proof of Theorem 1.

By *Lemma 1* we know now that the sum of prime factors of any given composite number greater than four is less than that number. This means, that the procedure involving sums of prime factors from *Theorem 1* is a terminating one. So:

$n < m < s < ... < t.$

The composite numbers must always be less and less after each sum of their prime factors; they terminate. This means, after t composite numbers, we must end up with a prime number. If this were false, then the procedure must end in a composite number, which cannot be true by *Lemma 1.* A composite number greater than four has always prime factors which can be summed up, resulting in a number less than the composite number itself, so it can never be the end of the procedure.
Therefore, given any composite number n being greater than four, the procedure demonstrated by *Theorem 1* must always terminate into a prime number greater than three. This is not the end of the proof yet. We must prove this to be true for every prime number greater than three. We achieve this with the help of prime friends.

Definition.

A *prime friend*[1] is a composite number whose sum of prime factors is equal to a prime number. It is expressed mathematically like so:

$$n=\prod_{i=1}^{k} p_i.$$

n is a prime friend if and only if

$$\sum_{i=1}^{k} p_i = p.$$

Examples:
6 is a prime friend of 5.
$6 = 2 \cdot 3$
$\Rightarrow 2 + 3 = 5$

12 is a prime friend of 7.
$12 = 2 \cdot 2 \cdot 3$
$\Rightarrow 2 + 2 + 3 = 7$

28 is a prime friend of 11.
$28 = 2 \cdot 2 \cdot 7$
$\Rightarrow 2 + 2 + 7 = 11$

Lemma 2.

1. There are infinitely many prime friends.

2. Every prime number after 3 has finitely many prime friends.

Proof:

We inspect the function $2k + 3$, where $k \in N$.

The difference of two prime numbers greater than two is always a multiple of two. This means that $2k + 3$ will generate every prime number greater than 2 with the appropriate input of k (for example: k cannot be a multiple of three since $2k + 3$ would be divisible by three). With this we can break up every prime number into a summand of 3 and k summands of 2.
2 and 3 are prime numbers, which means that this procedure of breaking up the sum implies that every prime number greater than three has at least one prime friend, since we can write the sum of k twos and a three like so:

$2 \cdot 1 + 3 = 2 + 3 = 5$
$\Rightarrow 2 \cdot 3 = 6$

$2 \cdot 2 + 3 = 4 + 3 = 2 + 2 + 3 = 7$
$\Rightarrow 2 \cdot 2 \cdot 3 = 12$

$2 \cdot 4 + 3 = 8 + 3 = 2 + 2 + 2 + 2 + 3 = 11$
$\Rightarrow 2 \cdot 2 \cdot 2 \cdot 2 \cdot 3 = 48$
.
.
.

A prime number has finitely many partitions, which also means that it cannot be expressed as a sum of prime numbers in an infinite amount of ways. Hence, with consideration of our previous argument, every prime number greater than three must have finitely many prime friends, meaning there must also be infinitely many prime friends overall. □

Now that we know that every prime number greater than three has finitely many prime friends, every other composite number must terminate into a prime number, which is given by *Lemma 1*.
Thus, every composite number greater than 4 will terminate into a prime number such that no prime number after 3 is left out in the process. $Q.E.D.$

3 Afterword.

To our knowledge, this field of number theory is greatly unknown. Researching this concept might extend our encyclopedia of facts about numbers for a few more pages. We encourage any readers to perhaps study this part of mathematics or anything related to it even further. One interesting fact for example might or might not help in proving the Twin prime conjecture:

Let us inspect the composite number p^p,
where p is a prime number greater than two.
By the use of *Theorem 1*, we know that p^p ends up as $p + 2$, where for certain values of p the composite number will terminate into a prime number of the form $p + 2$.

1. The word "prime friend" is something we have invented ourselves; it is not part of the standard mathematics vocabulary.