Seven Steps around the Perfect Cuboid

Zakhar Pekhterev

Proof of the impossibility of the existence of a perfect cuboid.

Step 1.

In mathematics, the *Perfect Cuboid* is a rectangular cuboid whose edges, face diagonals and space diagonal all have integer lengths: https://en.wikipedia.org/wiki/Euler_brick

A Perfect Cuboid must satisfy the following system of diophantine equations:

$$\begin{cases} a^2+b^2=d^2 \\ a^2+c^2=e^2 \\ b^2+c^2=f^2 \\ a^2+b^2+c^2=g^2 \end{cases} \qquad \text{(I)}$$

where: a,b,c are the edges, d,e,f are the face diagonals and g is the space diagonal.

Until now, there was no confirmation of the existence of a Perfect Cuboid, but it had not been proven either that such a cuboid cannot exist, so this has remained a problem for several centuries. However, I will try to show the reason why the Perfect Cuboid is impossible.

Step 2.

If Perfect cuboid exists, the squares of 3 its face diagonals should construct a Heronian triangle.

Proof.

From simple transformations of (I) we have:

$$2(a^2+b^2+c^2)=d^2+e^2+f^2 \qquad \text{(II)}$$

$$2g^2=d^2+e^2+f^2 \qquad \text{(III)}$$

$$g^2=\frac{d^2+e^2+f^2}{2} \qquad \text{(IV)}$$

$$\begin{cases} a^2=g^2-f^2 \\ b^2=g^2-e^2 \\ c^2=g^2-d^2 \end{cases} \qquad \text{(V)}$$

$$\begin{cases} a^2=\dfrac{d^2+e^2+f^2}{2}-f^2 \\ b^2=\dfrac{d^2+e^2+f^2}{2}-e^2 \\ c^2=\dfrac{d^2+e^2+f^2}{2}-d^2 \end{cases} \qquad \text{(VI)}$$

By substitution from (IV) and (VI) it is follows:

$$abcg=\sqrt{\left(\frac{d^2+e^2+f^2}{2}-f^2\right)\left(\frac{d^2+e^2+f^2}{2}-e^2\right)\left(\frac{d^2+e^2+f^2}{2}-d^2\right)\left(\frac{d^2+e^2+f^2}{2}\right)} \qquad \text{(VII)}$$

$$abcg=\frac{1}{4}\sqrt{\left(-d^2+e^2+f^2\right)\left(d^2-e^2+f^2\right)\left(d^2+e^2-f^2\right)\left(d^2+e^2+f^2\right)} \qquad \text{(VII)}$$

$abcg\in\mathbb{N}\Rightarrow d^2,e^2,f^2$ should be the sides of a Heronian triangle with square sides and an area equal to $abcg$ (https://en.wikipedia.org/wiki/Heronian_triangle). What was required.

Step 3.

Exact formula for all Heronian triangles.

The well-known parametric solution shows that every Heronian triangle has sides proportional to:

$$\begin{cases} a=n(m^2+k^2) \\ b=m(n^2+k^2) \\ c=(m+n)(mn-k^2) \end{cases}$$

where a,b,c are it's edges, $\gcd(m,n,k)=1, mn>k^2\geq 1, m\geq n\geq 1$.

Thus, by substitution from Step 2 our Heronian triangle with square sides will be similarly parametrized:

$$\begin{cases} d^2=n(m^2+k^2) \\ e^2=m(n^2+k^2) \\ f^2=(m+n)(mn-k^2) \end{cases} \qquad \text{(VIII)}$$

Step 4.

Now we can also express the rest of the parameters of the Perfect Cuboid. From (IV), (V) and (VIII) it is follows:

$$\begin{cases} g^2=\dfrac{d^2+e^2+f^2}{2}=mn(m+n) \\ a^2=g^2-f^2=k^2(m+n) \\ b^2=g^2-e^2=m(mn-k^2) \\ c^2=g^2-d^2=n(mn-k^2) \end{cases} \qquad \text{(IX)}$$

Step 5.

Parity table.

m	n	k	a	b	c	d	e	f	g
odd	odd	odd	even	even	even	even	even	even	even
odd	odd	even	even	odd	odd	odd	odd	even	even
odd	even	odd	odd	odd	even	even	odd	odd	even
odd	even	even	even	even	even	even	even	even	even
even	odd	odd	odd	even	odd	odd	even	odd	even
even	odd	even	even	even	even	even	even	even	even
even	even	odd	even	even	even	even	even	even	even
even	even	even	even	even	even	even	even	even	even

Step 6.

Using well-known property that must be satisfied by a primitive perfect cuboid:

- *One edge, two face diagonals and the body diagonal must be odd, one edge and the remaining face diagonal must be divisible by 4, and the remaining edge must be divisible by 16.*

According to the parity table from Step 5, it becomes clear that such cuboid is absent. Truly, this is proof of the impossibility of existence of Perfect Cuboid.

Step 7.

From (VIII) and (IX) let's define the squares sequentially:

$$a^2 = k^2(m+n) \Rightarrow (m+n) = \square$$

$$f^2 = (m+n)(mn-k^2) \Rightarrow (mn-k^2) = \square$$

$$b^2 = m(mn-k^2) \Rightarrow m = \square$$

$$c^2 = n(mn-k^2) \Rightarrow n = \square$$

In addition there is one more also well-known property:

- *The space diagonal can only contain prime divisors ≡ 1(mod 4).*

At the same time there is well-known fact that:

- *An exact square is either divisible by 4, or when divided by 8 gives a remainder of 1.*

Thus, we can write:

$$\begin{cases} m = \square = 8p+1 \\ n = \square = 8q+1 \end{cases}$$

but from (IX) it is follows: $g^2 = mn(m+n) = (8p+1)(8q+1)(8p+8q+2)$ (X)

Contradiction.

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