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Assuming $c < rad^2(abc)$ and The Beal's Conjecture Hold, Then The abc Conjecture Is False

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To the memory of my Father who taught me arithmetic
To my wife Wahida, my daughter Sinda and my son Mohamed Mazen

Abstract

In this paper, assuming that the conjecture $c < rad^2(abc)$ and Beal's Conjecture hold, I give, using elementary logic, the proof that the abc conjecture is false.

1. *Introduction*

Let a positive integer $a = \prod_i a_i^{\alpha_i}$, a_i prime integers and $\alpha_i \geq 1$ positive integers. We call *radical* of a the integer $\prod_i a_i$ noted by $rad(a)$. Then a is written as :

$$a = \prod_i a_i^{\alpha_i} = rad(a).\prod_i a_i^{\alpha_i - 1} \tag{1.1}$$

We note:

$$\mu_a = \prod_i a_i^{\alpha_i - 1} \Longrightarrow a = \mu_a.rad(a) \tag{1.2}$$

The abc conjecture was proposed independently in 1985 by David Masser of the University of Basel and Joseph Œsterlé of Pierre et Marie Curie University (Paris 6) [**1**]. It describes the distribution of the prime factors of two integers with those of its sum. The definition of the abc conjecture is given below:

Conjecture 1. For each $\epsilon > 0$, there exists $K(\epsilon)$ such that if a, b, c positive integers relatively prime with $c = a + b$, then :

$$c < K(\epsilon).rad^{1+\epsilon}(abc) \tag{1.3}$$

where K is a constant depending only of ϵ.

The idea to try to write a paper about this conjecture was born after the publication of an article in Quanta magazine, in November 2018, about the remarks of professors Peter Scholze of the University of Bonn and Jakob Stix of Goethe University Frankfurt concerning the proof of Shinichi Mochizuki [**2**]. The difficulty to find a proof of the abc conjecture is due to the incomprehensibility how the prime factors are organized in c giving a, b with $c = a + b$.

We know that numerically, $\dfrac{Logc}{Log(rad(abc))} \leq 1.629912$ [**1**]. A conjecture was proposed that $c < rad^2(abc)$: [**3**].

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Conjecture 2. Let a, b, c positive integers relatively prime with $c = a + b$, then:

$$c < rad^2(abc) \Longrightarrow \frac{Logc}{Log(rad(abc))} < 2 \tag{1.4}$$

After studying the *abc* conjecture using different choices of the constant $K(\epsilon)$ and having attacked the problem from diverse angles, I have arrived to conclude that *abc* conjecture, assuming that $c < rad^2(abc)$ and the Beal's conjecture hold, does not hold when $0 < \epsilon < 1$ in the case $R < c$. Then the *abc* conjecture as it was defined is false. In this note, I give the justifications that the *abc* conjecture is not true.

The paper is organized as follows: the second section is for the preliminaries needed to the proof. In section three, we give the proof that the *abc* conjecture is false.

2. *Preliminaries*

In 1997, Andrew Beal [**5**] announced the following conjecture :

Conjecture 3. *Let $A, B, C, m, n,$ and l be positive integers with $m, n, l > 2$. If:*

$$A^m + B^n = C^l \tag{2.1}$$

then $A, B,$ and C have a common factor.

In the following, we assume that the Beal's conjecture is true. Let $c = a + b$ with $(a, b, c) = 1$. From the Beal's conjecture, it follows that c can not be written as $c = C^l = a + b$ with $a = A^m$, $b = B^n$ and m, n, l large integers so that $R = rad(abc) \ll c$. Then the following remark:

Remark 1. Assuming the Beal's conjecture true, if $c = a + b$ with $(a, b, c) = 1$, is a very large integer then $R = rad(abc)$ is also a very large integer.

We recall the following proposition [**4**]:

Proposition 2.1. *Let $\epsilon \longrightarrow K(\epsilon)$ the application verifying the abc conjecture, then:*

$$lim_{\epsilon \to 0} K(\epsilon) = +\infty \tag{2.2}$$

3. *The Proof that the abc conjecture is false*

In the case $c < R$, for $\forall \epsilon > 0$, if we choose $K(\epsilon) = 1 + \epsilon$, then $c < K(\epsilon)R^{1+\epsilon}$ and the *abc* conjecture is true. Now, we consider in the following $R < c$ and we assume that the conjecture $c < R^2$ is true. I will give the proof that the *abc* conjecture does not hold when $0 < \epsilon < 1$ and $R < c$, then it follows that the *abc* conjecture is false.

Proof. - We recall the definition of the *abc* conjecture:
For each $\epsilon > 0$, there exists $K(\epsilon)$ such that if a, b, c positive integers relatively prime with $c = a + b$, then :

$$c < K(\epsilon).rad^{1+\epsilon}(abc) \tag{3.1}$$

where K is a constant depending only of ϵ.

From the equation (3.1) above, $K(\epsilon) > 0$.
Case-1- For $\epsilon = 1$, we can take $K(1) = 1$ and let a, b, c positive integers relatively prime with $c = a + b$, then we can write:

$$c < R^2 \Longrightarrow c < 1.R^{1+1} \tag{3.2}$$

and the *abc* conjecture is true for $\epsilon = 1$.

Case-2- For $\epsilon > 1$, we can choose $K(\epsilon) = \epsilon$. From $c < R^2$, we obtain :

$$c < R^2 < \epsilon.R^{1+1} < K(\epsilon).R^{1+\epsilon} \tag{3.3}$$

and the *abc* conjecture is true for $\epsilon > 1$.

Case-3- We consider $\epsilon \in]0, 1[$ and not near 1^-. We suppose that the *abc* conjecture is true, there exists $K(\epsilon)$ so that giving any c=a+b and $(a, b, c) = 1$, we have:

$$c < K(\epsilon)R^{1+\epsilon} \tag{3.4}$$

Now we will compare R^2 and $K(\epsilon)R^{1+\epsilon}$, there are three cases:

Case-3-1- If $R^2 < K(\epsilon)R^{1+\epsilon} \Longrightarrow LogR < \dfrac{LogK(\epsilon)}{1-\epsilon}$. But there is an obstruction to the inequality when we choose the triplet (a, b, c) so that $c = a + b$ is a very large integer and using the remark above, then R is very large. As ϵ is not near 1^-, then $\dfrac{LogK(\epsilon)}{1-\epsilon}$ is bounded, it follows the contradiction.

Case-3-2- If $R^2 = K(\epsilon)R^{1+\epsilon} \Longrightarrow R^{1-\epsilon} = K(\epsilon)$. Then the contradiction with $K(\epsilon)$ is a constant depending only of ϵ.

Case-3-3- If $K(\epsilon)R^{1+\epsilon} < R^2 \Longrightarrow K(\epsilon) < R^{1-\epsilon}$. If ϵ is chosen very small, and using the proposition 2.1 above, we obtain again a contradiction.

Hence the *abc* conjecture is not true for $\epsilon \in]0, 1[$.
We conclude that the *abc* conjecture is false and the proof is finished. □

4. *Conclusion*

Assuming the conjecture $c < R^2$ true as well as the Beal's conjecture, we have obtained by an elementary proof that the *abc* conjecture is false. We can announce the important theorem:

THEOREM 4.1. *Assuming the conjecture $c < R^2$ and the Beal's conjecture hold, then the abc conjecture is false.*

Alas! many problems in number theory remain open.

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