The Irrationality of Odd and Even Zeta Values

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Abstract

We show that using the denominators of the terms of $\zeta(n)-1 = z_n$ as decimal bases gives all rational numbers in (0,1) as single decimals. We also show the partial sums of z_n are not given by such single digits using the partial sum's terms. These two properties yield a proof that z_n is irrational. As partials require denominators exceeding the denominators of their terms, possible single decimal convergence points are, using properties of decimal expansions, systematically eliminated.

1 Introduction

Apery's $\zeta(3)$ is irrational proof [1] and its simplifications [3, 8] are the only proofs that a specific odd argument for $\zeta(n)$ is irrational. The irrationality of even arguments of zeta are a natural consequence of Euler's formula [2]:

$$\zeta(2n) = \sum_{k=1}^{\infty} \frac{1}{k^{2n}} = (-1)^{n-1} \frac{2^{2n-1}}{(2n!)} B_{2n} \pi^{2n}. \tag{1}$$

Apery also showed $\zeta(2)$ is irrational, and Beukers, based on the work (tangentially) of Apery, simplified both proofs. He replaced Apery's mysterious recursive relationships with multiple integrals. See Poorten [10] for the history of Apery's proof; Havil [5] gives an overview of Apery's ideas and attempts to demystify them. Also of interest is Huylebrouck's [6] paper giving an historical context for the main technique used by Beukers.

Attempts to generalize the techniques of the one odd success seem to be hopelessly elusive. Apery's and other ideas can be seen in the work of Rivoal and Zudilin [11, 12]. Their results, that there are an infinite number of odd n such that $\zeta(n)$ is irrational and at least one of the cases 5,7,9, 11 likewise irrational do suggest a radically different approach is necessary.

Let

$$z_n = \zeta(n) - 1 = \sum_{j=2}^{\infty} \frac{1}{j^n} \text{ and } s_k^n = \sum_{j=2}^{k} \frac{1}{j^n}.$$

We show that every rational number in $(0, 1)$ can be written as a single decimal using the denominators of a term in z_n as a number basis. But the partial sums can't be expressed with such a single decimal using the denominators of its terms as number bases. These two properties yield a proof that all z_n are irrational.

Properties of z_n

We define a decimal set.

Definition 1. *Let*

$$d_{j^n} = \{1/j^n, \ldots, (j^n - 1)/j^n\} = \{.1, \ldots, .(j^n - 1)\} \textit{ base } j^n.$$

That is d_{j^n} consists of all single decimals greater than 0 and less than 1 *in base j^n. The decimal set for j^n is*

$$D_{j^n} = d_{j^n} \setminus \bigcup_{k=2}^{j-1} d_{k^n}.$$

The set subtraction removes duplicate values.

Definition 2.

$$\bigcup_{j=2}^{k} D_{j^n} = \Xi_k^n$$

The union of decimal sets gives all rational numbers in $(0, 1)$.

Lemma 1.

$$\bigcup_{j=2}^{\infty} D_{j^n} = \mathbb{Q}(0, 1)$$

Proof. Every rational $a/b \in (0,1)$ is included in a d_{b^n} and hence in some D_{r^n} with $r \leq b$. This follows as $ab^{n-1}/b^n = a/b$ and as $a < b$, per $a/b \in (0,1)$, $ab^{n-1} < b^n$ and so $a/b \in d_{b^n}$. $\square$

Next we show $s_k^n \notin \Xi_k^n$; that is: we show that partial sums of z_n can't be expressed as a single decimal using number bases given by the denominators of the partial's terms.

Lemma 2. *If $s_k^n = r/s$ with r/s a reduced fraction, then 2^n divides s.*

Proof. The set $\{2, 3, \ldots, k\}$ will have a greatest power of 2 in it, a; the set $\{2^n, 3^n, \ldots, k^n\}$ will have a greatest power of 2, na. Also $k!$ will have a powers of 2 divisor with exponent b; and $(k!)^n$ will have a greatest power of 2 exponent of nb. Consider

$$\frac{(k!)^n}{(k!)^n} \sum_{j=2}^{k} \frac{1}{j^n} = \frac{(k!)^n/2^n + (k!)^n/3^n + \cdots + (k!)^n/k^n}{(k!)^n}. \tag{2}$$

The term $(k!)^n/2^{na}$ will pull out the most 2 powers of any term, leaving a term with an exponent of $nb - na$ for 2. As all other terms but this term will have more than an exponent of 2^{nb-na} in their prime factorization, we have the numerator of (2) has the form

$$2^{nb-na}(2A + B),$$

where $2 \nmid B$ and A is some positive integer. This follows as all the terms in the factored numerator have powers of 2 in them except the factored term $(k!)^n/2^{na}$. The denominator, meanwhile, has the factored form

$$2^{nb}C,$$

where $2 \nmid C$. This leaves 2^{na} as a factor in the denominator with no powers of 2 in the numerator, as needed. $\square$

Lemma 3. *If $s_k^n = r/s$ with r/s a reduced fraction and p is a prime such that $k > p > k/2$, then p^n divides s.*

Proof. First note that $(k, p) = 1$. If $p|k$ then there would have to exist r such that $rp = k$, but by $k > p > k/2$, $2p > k$ making the existence of such a natural number $r > 1$ impossible.

The reasoning is much the same as in Lemma 2; cf. Chapter 2, Problem 21 in [2], solution in [7]. Consider

$$\frac{(k!)^n}{(k!)^n}\sum_{j=2}^{k}\frac{1}{j^n}=\frac{(k!)^n/2^n+\cdots+(k!)^n/p^n+\cdots+(k!)^n/k^n}{(k!)^n}. \tag{3}$$

As $(k,p)=1$, only the term $(k!)^n/p^n$ will not have p in it. The sum of all such terms will not be divisible by p, otherwise p would divide $(k!)^n/p^n$. As $p<k$, p^n divides $(k!)^n$, the denominator of r/s, as needed. □

Lemma 4. *For any $k \geq 2$, there exists a prime p such that $k<p<2k$.*

Proof. This is Bertrand's postulate [4]. □

Theorem 1. *If $s_k^n=\frac{r}{s}$, with r/s reduced, then $s>k^n$.*

Proof. Using Lemma 4, for even k, we are assured that there exists a prime p such that $k>p>k/2$. If k is odd, $k-1$ is even and we are assured of the existence of prime p such that $k-1>p>(k-1)/2$. As $k-1$ is even, $p \neq k-1$ and $p>(k-1)/2$ assures us that $2p>k$, as $2p=k$ implies k is even, a contradiction.

For both odd and even k, using Lemma 4, we have assurance of the existence of a p that satisfies Lemma 3. Using Lemmas 2 and 3, we have $2^n p^n$ divides the denominator of r/s and as $2^n p^n > k^n$, the proof is completed. □

Corollary 1.

$$s_k^n \notin \Xi_k^n$$

Proof. This is a restatement of Theorem 1. □

Corollary 2. *For sufficiently large k decimal representations of s_k^n in any base b^n must be of the mixed or pure repeating variety.*

Proof. As the prime factors in the denominators of s_k^n grow without bound with increasing k (per Lemma 4), for a fixed base b it must be that only some, possible no prime factors of b are shared with the reduced fraction denominators of s_k^n. If some are shared the resulting decimal expressions is mixed; if none are shared then the resulting decimal is pure repeating [4]. □

Definition 3. *A mixed decimal $.h_1\dots h_d\overline{t_1\dots t_l}$ is said to have a head length of d and a period of l. A pure repeating decimal $.\overline{p_1\dots p_m}$ is said to have a period of m.*

Corollary 3. *The expressions of s_k^n in base b^n have lengths and periods that increase without bound.*

Proof. As the reduced denominators of s_k^n have changing prime factors, given the uniqueness of representations of such fractions, if lengths and periods were fixed, they would be exhausted given the infinity of prime factors. $\square$

Corollary 4. *In any base b^n, for sufficiently large k the representation of s_k^n can't be of the form $.(a-1)\overline{(b-1)}$.*

Proof. The decimals making up mixed and pure decimals expansions in a given base do not have repeating parts consisting of all nines – referencing base 10 numbers. This follows from Fermat's theorem that gives the periods of pure repeating and mixed decimals in base b [4]: the ν in

$$b^{\nu} \equiv 1 \bmod d, \tag{4}$$

where c/d is the fraction represented. The digits are given by the x that (4) implies exist. There must be an x such that

$$b^{\nu} - 1 = xd. \tag{5}$$

But (5) implies that

$$\frac{x}{b^{\nu}-1} = \frac{1}{d}.$$

As $b^{\nu} - 1$ is a string of ν $(b-1)$s, x can't be such a string. $\square$

Example 1. Consider $1/7$ in base 10. For this number $10^6 \equiv 1 \bmod 7$ (Fermat's theorem) and there must exist an x such that

$$10^6 - 1 = x7 \text{ or } \frac{x}{10^6-1} = \frac{1}{7}.$$

In this case x is 142857 and $.\overline{142857} = 1/7$. We see that x can never equal ν nines, as the resulting fraction is 1, not a fraction. Note: the base of interest here is 10^6 and for this base the equivalent of 9 in base 10 is $999999 = 10^6 - 1$, ν nines.

Example 2. Note: The number $.\overline{099}$ is really $.(099)_{10^3}$, that is a repeating single digit, a repeating symbol within the 999 symbols of base 1000; the symbol for the equivalent of 9 in this base is 999.

Example 3. For mixed decimals, consider 1/6 base 10. We have

$$10\frac{1}{2 \cdot 3} = \frac{5}{3} = 1 + \frac{2}{3} = 1 + .\overline{6}.$$

Notice that now we have a whole number on the far right side plus a reduced fraction with a denominator that is relatively prime to 10. This pattern will always occur in fractions with denominators that share some but not all the prime divisors with the given base. Now we can divide by 10 and arrive at

$$\frac{1}{6} = \frac{1}{10} + \frac{.\overline{6}}{10} = .1\overline{6},$$

as expected. All mixed fractions will have a pure repeating part possessed of a period of finite length. We use this property in our main proof.

z_n is irrational

Theorem 2. *z_n is irrational.*

Proof. Without loss of generality, per Lemma 1, assume that $z_n = .x$ in base 10. This implies that $z_n = .(x-1)\overline{9}$, but per Corollaries 2, 3, and 4 this is impossible. $\square$

Conclusion

Finally, this result surviving public scrutiny, there is the possibility of its relevance to the premier number theory open problem: the Riemann hypotheses. I have some hope that the equivalent of number bases (plural) in the complex number system might allow the same exclusions used here (irrational not rational) to carry over to a zero versus not a zero. There are Gaussian integers and Gaussian primes; might there be forms of number bases that inform us of the location of zeros for the ever wonderful and mysterious zeta function.

References

[1] Apéry, R. (1979). Irrationalité de $\zeta(2)$ et $\zeta(3)$. *Astérisque* 61: 11-13.

[2] Apostol, T. M. (1976). *Introduction to Analytic Number Theory.* New York: Springer.

[3] Beukers, F. (1979). A Note on the irrationality of $\zeta(2)$ and $\zeta(3)$, *Bull. London Math. Soc.* 11: 268–272.

[4] Hardy, G. H., Wright, E. M., Heath-Brown, R. , Silverman, J. , Wiles, A. (2008). *An Introduction to the Theory of Numbers*, 6th ed. London: Oxford Univ. Press.

[5] J. Havil (2012). *The Irrationals.* Princeton, NJ: Princeton Univ. Press.

[6] Huylebrouck, D. (2001). Similarities in irrationality proofs for π, $\ln 2$, $\zeta(2)$, and $\zeta(3)$, *Amer. Math. Monthly* 108(10): 222–231.

[7] Hurst, G. (2014). Solutions to Introduction to Analytic Number Theory by Tom M. Apostol. https://greghurst.files.wordpress.com/2014/02/apostol_intro_to_ant.pdf

[8] Nesterenko, Y. V. (1996). A few remarks on $\zeta(3)$, *Math. Zametki* 59(6): 865–880.

[9] Niven, I. (2006). Irrational Numbers. MAA.

[10] van der Poorten, A. (1978/9). A proof that Euler missed... Apery's proof of the irrationality of $\zeta(3)$, an informal report. *Math. Intelligencer* 1(4): 195–203.

[11] Rivoal, T. (2000). La fonction zeta de Riemann prend une infinit de valeurs irrationnelles aux entiers impairs, *Comptes Rendus de l'Acadmie des Sciences, Srie I. Mathmatique* 331: 267-270.

[12] Zudilin, W. W. (2001). One of the numbers $\zeta(5)$, $\zeta(7)$, $\zeta(9)$, $\zeta(11)$ is irrational. *Russian Mathematical Surveys* 56(4): 747–776.