Improvement of Prime Number Theorem using the Multi-Point Summation Method

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Abstract

I propose a new approximate asymptotic formula for Prime number theorem. The new formula is derived by the multi-point summation method. It has additional term expressed with elementary function and gives better estimate of the prime-counting function from small value to big value. It also satisfies asymptotic formula with $n \to \infty$ limit.

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The prime number theorem is a theorem that describes the asymptotic behavior of a prime-counting function $\pi(n)$ that represents the number of prime numbers less than or equal to number n. It is known that asymptotic behavior can be described by $\pi(n) = \mathrm{li}(n) - \mathrm{li}(2)$ for a large number n[1]. However, this formula has a large deviation in value for small number n. The multi-point summation method(MPS)[2] is a summation method that gives an approximate function that perfectly reproduces the asymptotic behavior between two points including the point at infinity. The advantage of this method is that it can improve the behavior from a small number n without changing the asymptotic behavior at infinity n.

I derive asymptotic series for small number n of $\pi(n)$ by fitting for $n \leq 50$ around $n = 10$. I take logarithmic scale for n.

$$\pi(n) \sim f_0(n) \equiv 0.487(\ln n - \ln 10)^3 + 0.864(\ln n - \ln 10)^2 + 2.473(\ln n - \ln 10) - 1.200 \quad (1)$$

This asymptotic series is valid only for small number $\ln n \sim 2$.

For big number n, asymptotic series of $\pi(n)$ is given by[1],

$$\pi(n) \sim \mathrm{li}(n) - \mathrm{li}(2) \quad (2)$$

I rescale number n by logarithm. I get,

$$\pi(\exp(n)) \sim \mathrm{li}(\exp(n)) - \mathrm{li}(2) \quad (3)$$

Expanding around $n = 10$,

$$\mathrm{li}(\exp(n)) - \mathrm{li}(2) \equiv f_\infty(n) \sim 0.368(\ln n)^3 - 1.315(\ln n)^2 + 4.542 \ln n - 2.861 \quad (4)$$

The multi-points summation method uses both asymptotic series in Eqns.(1) and (3) By subracting asymptotic terms $f_\infty(n)$ at infinite from $f_0(n)$, I obtain

$$g(n) = f_0(n) - f_\infty(n) = -0.096(\ln n)^3 + 1.046(\ln n)^2 - 4.246 \ln n + 4.501 \quad (5)$$

I take [1/2] Padé approximant here,

$$g(n) \simeq \frac{2.037 - 1.274 \ln n}{0.413 + 0.148 \ln n + 0.047(\ln n)^2} \quad (6)$$

Then, I get final approximate expression with correct asymptotic series at $n \to \infty$,

$$\pi(n) \simeq \mathrm{li}(n) - \mathrm{li}(2) + \frac{2.037 - 1.274 \ln n}{0.413 + 0.148 \ln n + 0.047(\ln n)^2} \quad (7)$$

This formula is consistent with asymptotic series at two limit of $\ln n \to 2$ and $n \to \infty$. In this respect, MPS is different from just fitting. For the sake of readability, it should be mentioned that the numbers are rounded to 4 significant digits.

I plotted obtained expressions in Fig.1 and Fig.2. $\pi(n)$ is plotted with blue curve. Eq.(**??**) which is result of MPS is plotted with red curve. Eq.(1) is plotted with green curve. Eq.(**??**) is plotted with black curve.

The values of the ratio between $\pi(n)$ and each formulas is shown in Table.**??**.

MPS gives better estimates for all range of n than just logarithmic integral.

I perform the MPS with asymptotic series with small number and infinitely big number. Obtained formula is simple as expressed additional elementary function and gives better estimates of the prime-counting function $\pi(n)$.

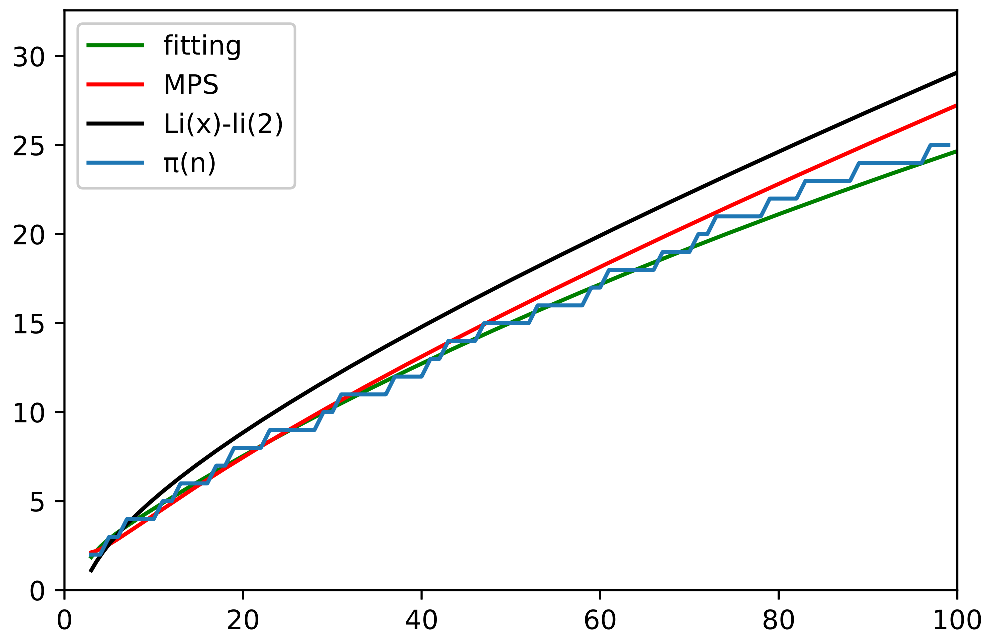

Figure 1: The number of primes with some asymptotic formulas between $n = 1 \sim 100$.

References

[1] Y. Kida: *Elementary number theory* (Asakura Shoten, 2001).

[2] Y. Ueoka: *Introduction to multipoints summation method* (Kindle, 2020).

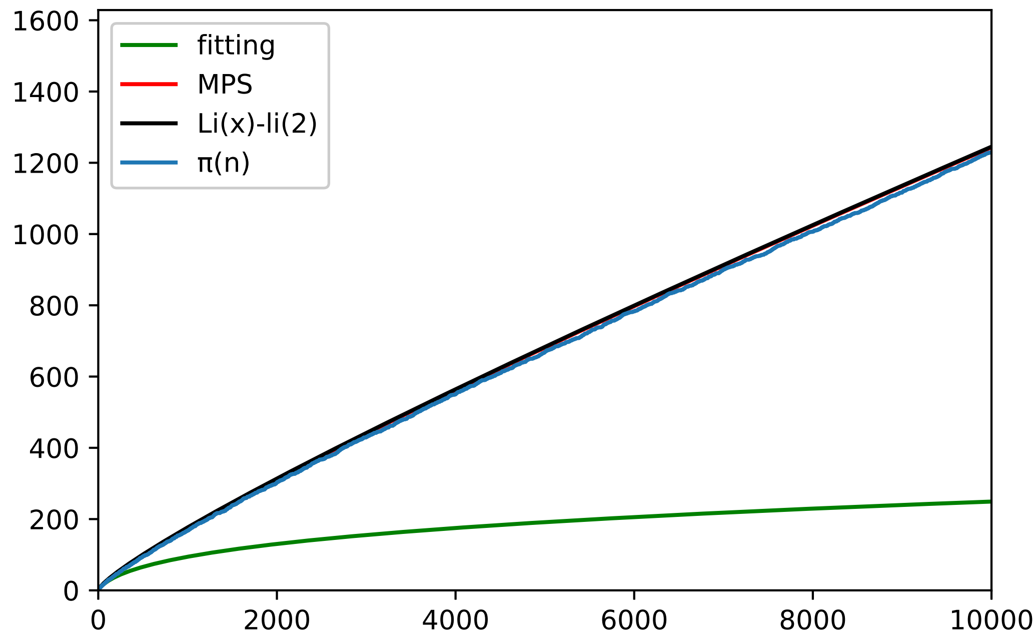

Figure 2: The number of primes with some asymptotic formulas between $n = 1 \sim 10000$.

n	$\pi(n)$/Eq.(7)	$\pi(n)$/Eq.(**??**)
5	1.17	1.16
10	0.95	0.78
50	0.96	0.86
100	0.91	0.86
500	0.96	0.94
1000	0.96	0.95
10^4	0.99	0.99

Table 1: Approximation formula for the prime-counting function $\pi(n)$.