THE SPANNING METHOD AND THE LEHMER TOTIENT PROBLEM

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Abstract. In this paper we introduce and develop the notion of spanning of integers along functions $f : \mathbb{N} \longrightarrow \mathbb{R}$. We apply this method to a class of problems requiring to determine if the equations of the form $tf(n) = n - k$ has a solution $n \in \mathbb{N}$ for a fixed $k \in \mathbb{N}$ and some $t \in \mathbb{N}$. In particular, we show that

$$\#\{n \leq s \mid t\varphi(n) + 1 = n,\ t, n \in \mathbb{N}\} \geq \frac{s}{2\log s}\prod_{p|s}(1-\frac{1}{p})^{-1} + O(1)$$

where φ is the euler totient function.

1. Introduction and problem statement

Let $\varphi : \mathbb{N} \longrightarrow \mathbb{N}$ be the euler totient function, so that $\varphi(s)$ is the number of integers $n \leq s$ and co-prime with s. The image of the euler totient function is defined on prime number arguments as the unit left translate of the primes; in particular, we have $\varphi(p) = p - 1$ and one can clearly see that $\varphi(p)|p - 1$. It is natural to speculate if composites also satisfy the divisibility relation. To this end, The mathematician D.H Lehmer posed the question which is now known as the **Lehmer totient problem**

Question 1.1. Can the totient function of a composite number n divide $n - 1$?

The euler totient problem is considerably of the same class as the odd perfect number problem. D.H Lehmer showed that if there exists such composite number n, then it must be odd, square-free and have at least **seven** distinct prime factors [2]. Further improvements were made by Hagis and Cohen in 1980, who showed that if such composite number n exists then it must satisfy $n \geq 10^{20}$ and have at least **fourteen** distinct prime factors [1]. This was further improved by Hagis proving that if 3 divides n, then $n \geq 10^{1937042}$ and having at least 298848 distinct prime factors [4]. It is also known (see [3]) that the number of solutions $\leq x$ to the Lehmer totient problem satisfy the upper bound

$$\leq \frac{\sqrt{x}}{(\log x)^{\frac{1}{2}+o(1)}}.$$

In this paper we study the Lehmer totient problem using the lower bound

Lemma 1.2. *The lower bound holds*

$$\#\{n \leq s \mid t\varphi(n) + 1 = n,\ t, n \in \mathbb{N}\} \geq \frac{s}{2\log s}\prod_{p|s}(1-\frac{1}{p})^{-1} + O(1)$$

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where φ is the euler totient function.

2. **Preliminary results**

In this paper, we find the following elementary inequalities useful. We will employ them in the course of establishing the main result of this paper.

Lemma 2.1. *Let $S(x)$ denotes the sum of all prime number $\leq x$. Then the inequality holds*

$$S(x) > \frac{x^2}{2\log x} + \frac{x^2}{4\log^2 x} + \frac{x^2}{4\log^3 x} + \frac{1.2x^2}{8\log^4 x}.$$

for all $x \geq 905238547$.

Proof. For a proof see for instance [5]. □

Lemma 2.2 (The prime number theorem)**.** *Let $\pi(x)$ denotes the number of primes $\leq x$. Then*

$$\pi(x) \sim \frac{x}{\log x}.$$

Lemma 2.3 (Merten's formula)**.** *The asymptotic holds*

$$\prod_{p\leq s}(1-\frac{1}{p}) \sim \frac{e^{-\gamma}}{\log s}$$

where $\gamma = 0.5772\cdots$ is the euler-macheroni constant.

Lemma 2.4 (Stieltjes-Lebesgue integral)**.** *Let $g : [a,b] \longrightarrow \mathbb{R}$ and $h : [a,b] \longrightarrow \mathbb{R}$ be right continuous and of bounded variation on $[a,b]$ and both having left limits. Then we have*

$$f(a)g(a) - f(b)g(b) = \int\limits_{(a,b]} f(t^-)dg(t) + \int\limits_{(a,b]} g(t^-)df(t) + \sum_{t\in(a,b]} \Delta f_t \Delta g_t$$

where $\Delta f_t = f(t) - f(t^-)$.

3. **The method of spanning along a function**

In this section we introduce and study the notion of **spanning** of integers along a function. We study this notion together with associated statistics and explore some applications.

Definition 3.1. Let $f : \mathbb{N} \longrightarrow \mathbb{R}$. Then we say n is k - step **spanned** along the function with multiplicity t if

$$tf(n) + k = n.$$

We call the set of all $n \in \mathbb{N}$ such that n is k - step spanned the k^{th} - step spanning set along f and denote by $\mathbb{S}_k(f)$. We call the set of all truncated k-step spanning set $\mathbb{S}_k(f) \cap \mathbb{N}_s := \mathbb{S}_k(f,s)$ the s^{th} scale spanned along f. We write the length of this spanned set as

$$|\mathbb{S}_k(f,s)| := \#\{n \leq s \mid tf(n) + k = n,\ t \in \mathbb{N}\}.$$

It is easy to see that $\mathbb{S}_k(f,s) < s$.

3.1. **The s-level measure of spanned set.** In this section we introduce the notion of the measure of the span set. We launch and examine the following languages.

Definition 3.2. By the s^{th} level measure of the span set $\mathbb{S}_k(f)$, denoted $\mathbb{M}_f(s,k)$, we mean the partial sum

$$\mathbb{M}_f(s,k) := \sum_{\substack{2\leq n\leq s \\ n\in\mathcal{S}_k(f)}} f(n).$$

Let us suppose that f is a right-continuous function and of bounded variation on $[j-1,j)$ for all $j\geq 3$ with $j\in\mathbb{N}$ and with a left limit, then by applying the Stieltjes-Lebesgue integration by parts, we can write the s^{th} level measure of the span set in the form

$$\begin{aligned}\mathbb{M}_f(s,k) :&= \sum_{2\leq j\leq s}\sum_{\substack{j-1<n\leq j \\ n\in\mathbb{S}_k(f)}} f(n) \\ &= \sum_{2\leq j\leq s}\int\limits_{(j-1)}^{j} f(t)d|\mathbb{S}_k(f,t)| \\ &< \sum_{2\leq j\leq s}\bigg(f(j)|\mathbb{S}_k(f,j)| - f(j-1)|\mathbb{S}_k(f,j-1)|\bigg) \\ &= f(s)|\mathbb{S}_k(f,s)| - f(1)|\mathbb{S}_k(f,1)|.\end{aligned}$$

The following inequality is a simple consequence of the above analysis.

Proposition 3.1 (Spanning inequality)**.** *Let f be a right-continuous function and of bounded variation on $[x,x+1)$ for $x\geq 1$ with $x\in\mathbb{N}$ and have left limits. Then the inequality holds*

$$|\mathbb{S}_k(f,s)| \geq \frac{1}{f(s)}\sum_{\substack{2\leq n\leq s \\ n\in\mathcal{S}_k(f)}} f(n) + \frac{f(1)|\mathbb{S}_k(f,1)|}{f(s)}.$$

4. **The fractional totient invariant function**

In this section we introduce and study a new function defined on the real line. We launch the following languages.

Definition 4.1. By the fractional totient invariant function, we mean the function $\tilde{\varphi} : [1,\infty) \longrightarrow \mathbb{R}$ such that

$$\tilde{\varphi}(a) = \varphi(\lfloor a\rfloor) + \{a\}$$

where φ is the euler totient function and $\lfloor\cdot\rfloor$ and $\{\cdot\}$ is the floor and the fractional part of a real number, respectively.

The fractional totient invariant function turns out to be an interesting function that in some way extends the euler totient function to the reals. Even though the notion of co-primality in not well-defined on the entire real line, it captures the intrinsic property of the euler totient function defined on the positive integers. In essence, the euler totient function and the fractional totient invariant function coincides on the set of positive integers. Next, we examine some elementary properties of the fractional totient invariant function in the following sequel.

Proposition 4.1. *The following properties of the fractional totient invariant function holds*

(i) *If a is a positive integer, then $\tilde{\varphi}(a) = \varphi(a)$.*
(ii) *$\tilde{\varphi}(a) < a$ for all $a > 1$.*

Remark 4.2. We now state an analytic property of the fractional totient invariant function. In fact, the fractional totient invariant function can be seen as a slightly continuous analogue of the euler totient function on subsets of the reals.

Proposition 4.2. *The function $\tilde{\varphi} : [1, \infty) \longrightarrow \mathbb{R}$ with*

$$\tilde{\varphi}(a) = \varphi(\lfloor a \rfloor) + \{a\}$$

is right-continuous and of bounded variation on $[x, x+1)$ for $x \geq 1$ with $x \in \mathbb{N}$ and have left limits.

5. **Main result**

Lemma 5.1. *The lower bound holds*

$$\#\{n \leq s \mid t\varphi(n) + 1 = n,\ t, n \in \mathbb{N}\} \geq \frac{s}{2\log s}\prod_{p|s}(1 - \frac{1}{p})^{-1} + O(1)$$

where φ is the euler totient function.

Proof. By appealing to Proposition 3.1, we obtain the lower bound

$$\#\{2 \leq n \leq s \mid t\tilde{\varphi}(n) + 1 = n,\ t, n \in \mathbb{N}\} \geq \frac{1}{\tilde{\varphi}(s)}\sum_{\substack{2 \leq n \leq s \\ n \in \mathbb{S}_1(\tilde{\varphi})}} \tilde{\varphi}(n) + O(1). \tag{5.1}$$

Next we estimate each term on the right-hand side of the inequality. Since $\varphi(p) = p - 1$ for any prime number $p \in \mathbb{P}$, we obtain the lower bound

$$\begin{aligned}\sum_{\substack{2 \leq n \leq s \\ n \in \mathbb{S}_1(\tilde{\varphi})}} \tilde{\varphi}(n) &\geq \sum_{p \leq s} \varphi(p) \\ &= \sum_{p \leq s} p - \pi(s).\end{aligned}$$

By applying Lemma 2.1, we obtain the lower bound for sufficiently large values of s

$$\sum_{p \leq s} p - \pi(s) \geq \frac{s^2}{2\log s} - \pi(s)$$

so that by appealing to the decomposition

$$\varphi(s) = s\prod_{p|s}(1 - \frac{1}{p})$$

with $\tilde{\varphi}(s) \sim \varphi(s)$ and Lemma 2.3, we obtain the lower bound

$$\frac{1}{\tilde{\varphi}(s)} \sum_{\substack{2 \le n \le s \\ n \in \mathbb{S}_1(\tilde{\varphi})}} \tilde{\varphi}(n) \ge \frac{s}{2\log s} \prod_{p|s} (1 - \frac{1}{p})^{-1} - \frac{1}{\tilde{\varphi}(s)} \pi(s). \tag{5.2}$$

By plugging (5.2) into (5.1) and applying the prime number theorem, we obtain the lower bound

$$\#\{2 \le n \le s \mid t\tilde{\varphi}(n) + 1 = n,\ t, n \in \mathbb{N}\} \ge \frac{s}{2\log s} \prod_{p|s} (1 - \frac{1}{p})^{-1} - \frac{\pi(s)}{\varphi(s)} + O(1)$$

and the claim inequality holds for $s \ge s_o$ since $\tilde{\varphi}(n) = \varphi(n)$ for each $n \in \mathbb{N}$. □

Theorem 5.2. *There exists a composite* $n \in \mathbb{N}$ *such that* $\varphi(n)|n-1$.

Proof. Suppose on the contrary that there exists no composite $n \in \mathbb{N}$ such that $\varphi(n)|n-1$. Then for all $s \ge s_o$, we obtain the lower bound by appealing to Lemma 5.1

$$\pi(s) \ge \frac{s}{2\log s} \prod_{p|s} (1 - \frac{1}{p})^{-1} + O(1)$$

where $\pi(s)$ is the prime counting function. Now let the prime $p_o \in \mathbb{P}$ be sufficiently large and choose

$$s := \prod_{p \le p_o} p$$

then it can be checked that

$$\prod_{p|s} (1 - \frac{1}{p})^{-1} \ge 3 \tag{5.3}$$

so that

$$\pi(s) \ge \frac{3}{2} \frac{s}{\log s} + O(1)$$

which contradicts the prime number theorem. □

References

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