Corrections about V. S. Adamchik's papers

Denis GALLET
Rectorat de DIJON
2 G, rue Général Delaborde 21000 DIJON (France)
densg71@gmail.com

March 21, 2022

Abstract

In this paper, I talk about some integrals and Melzak's product in terms of Barnes function but there are mistakes and so I give the real formulas.In total, there are three integrals and in the same time, I give more general formulas.
We find two papers:

-Contributions to the theory for the Barnes function for one integral. (1)

-Multiple Gamma Function and Its Application to Computation of Series and Products for two integrals and the Melzak's product. (2)

1 Definition

The Barnes function is defined as the following Weierstrass product:

$$G\left(1+z\right)=\left(2\,\pi\right)^{\frac{z}{2}}\,\mathrm{e}^{-\frac{z\left(1+z\right)}{2}-\frac{\gamma\,z^{2}}{2}}\prod_{k=1}^{\infty}\left(1+\frac{z}{k}\right)^{k}\mathrm{e}^{-z+\frac{z^{2}}{2\,k}} \qquad (3)$$

where gamma is the Euler-Mascheroni constant.

The following properties of G are well-known.

2 Properties

$$G(1) = 1 \quad (4)$$

$$G(1+z) = G(z)\,\Gamma(z) \quad (5)$$

$$\log(G(1+z)) = \frac{z\log(2\pi)}{2} - \frac{z(1+z)}{2} + z\log(\Gamma(1+z)) - \int_0^z \log(\Gamma(t+1))\,dt \quad (6)$$

$$\int_0^z \log(\Gamma(t+1))\,dt = \frac{z\log(2\pi)}{2} - \frac{z(1+z)}{2} + z\log(\Gamma(1+z)) - \log(G(z)) - \log(\Gamma(z)) \quad (7)$$

3 The first integral

$$\int_0^\infty \frac{x^2}{e^{2\pi x}-1}\arctan\left(\frac{x}{z}\right)dx$$

Let A be the Glaisher–Kinkelin's constant (8),K be the Catalan's constant (9).

Consider the integral $\int_0^\infty \frac{x^2}{e^{2\pi x}-1}\arctan\left(\frac{x}{z}\right)dx$

z is a positiv number.

So in the paper Contributions to the theory for the Barnes function page 12:

2log $(\Gamma_3(\mathrm{z}+1)) = -\frac{z^3(\log(z)-\mathrm{H}_3)}{2} + \log(G(1+z)) + z^2\zeta(1,0) - 2z\zeta(1,-1) + \zeta(1,-2) + 2\int_0^\infty \frac{x^2}{e^{2\pi x}-1}\arctan\left(\frac{x}{z}\right)dx$

There are 2 mistakes, it's difficult to explain and the real identity is:

-2log $(\Gamma_3(\mathrm{z}+1)) = -\frac{z^3(\log(z)-\mathrm{H}_3)}{3} + \log(G(1+z)) + z^2\zeta(1,0) - 2z\zeta(1,-1) + \zeta(1,-2) + 2\int_0^\infty \frac{x^2}{e^{2\pi x}-1}\arctan\left(\frac{x}{z}\right)dx$

Where H_3 is a harmonic number and we know that $\mathrm{H}_3 = 11/6$ and $\Gamma_3(\mathrm{z})$ is the triple gamma function.

It's possible to give the triple gamma function in terms of $\zeta(1,-2,z)$ where $\zeta(1,-2,z)$ is the derivative of the Hurwitz Zeta function at z.

So I have the formula of the integral $\int_0^\infty \frac{x^2}{e^{2\pi x}-1}\arctan\left(\frac{x}{z}\right)dx$

$-\frac{11\,z^3}{36}+\frac{z}{12}-\frac{\zeta(1,-2,z)}{2}+\frac{z^2\log(2\,\pi)}{4}+\frac{\log(\Gamma(z))z^2}{2}-\log\left(\Gamma\left(z\right)\right)z-$
$z\log\left(G\left(z\right)\right)-z\log\left(A\right)+\frac{z^3\log(z)}{6}$

It's possible to obtain a more general formula for $\int_0^\infty \frac{x^2}{\mathrm{e}^{a\pi\,x}-1}\arctan\left(\frac{x}{z}\right)\mathrm{d}x$

Where a and z both positiv number.

$-\frac{11\,z^3}{36}+\frac{z}{3\,a^2}-4\,\frac{\zeta\left(1,-2,\,\frac{az}{2}\right)}{a^3}+\frac{z^2\log(2\,\pi)}{2\,a}+\frac{z^2}{a}\log\left(\Gamma\left(\frac{az}{2}\right)\right)-4\,\frac{\log\left(\Gamma\left(\frac{az}{2}\right)\right)z}{a^2}-$
$4\,\frac{z\log\left(G\left(\frac{az}{2}\right)\right)}{a^2}-4\,\frac{z\log(A)}{a^2}+\frac{z^3}{6}\log\left(\frac{az}{2}\right)$

4 The second integral

$$\int_0^\infty \frac{x^2+z^2}{\mathrm{e}^{2\,\pi\,x}-1}\arctan\left(\frac{x}{z}\right)\cos\left(2\,\arctan\left(\frac{x}{z}\right)\right)\mathrm{d}x$$

z is a positiv number.

We find this integral in the paper Multiple Gamma Function and Its Application to Computation of Series and Products page 5.

There are a connection with the first integral.

Using the relationship, we have the formula of the integral:

$\frac{29\,z^3}{36}-\frac{z}{12}-\frac{2\,z^3\log(z)}{3}+\frac{z^2\log(z)}{4}-\frac{z^2\log(2\,\pi)}{2}+\log\left(\Gamma\left(z\right)\right)z+z\log\left(G\left(z\right)\right)+$
$z\log\left(A\right)+\frac{\zeta(1,-2,z)}{2}$

It's possible to obtain a more general formula for $\int_0^\infty \frac{x^2+z^2}{\mathrm{e}^{a\pi\,x}-1}\arctan\left(\frac{x}{z}\right)\cos\left(2\,\arctan\left(\frac{x}{z}\right)\right)\mathrm{d}x$

Where a and z both positiv number.

$\frac{29\,z^3}{36}-\frac{z}{3\,a^2}-\frac{2\,z^3}{3}\log\left(\frac{az}{2}\right)+\frac{z^2}{2\,a}\log\left(\frac{az}{2}\right)-\frac{z^2\log(2\,\pi)}{a}+4\,\frac{z\log\left(\Gamma\left(\frac{az}{2}\right)\right)}{a^2}+$
$4\,\frac{z\log\left(G\left(\frac{az}{2}\right)\right)}{a^2}+4\,\frac{z\log(A)}{a^2}+4\,\frac{\zeta\left(1,-2,\,\frac{az}{2}\right)}{a^3}$

5 The third integral

$$\int_0^\infty \frac{1}{e^{2\pi x}-1}\log\left(x^2+z^2\right)\sin\left(2\arctan\left(x/z\right)\right)\left(x^2+z^2\right)\mathrm{d}x$$

z is a positiv number.

We find this integral in the paper Multiple Gamma Function and Its Application to Computation of Series and Products page 5.

Using the relationship page 5, we have the formula of the integral:

$-\frac{3z^3}{2}+\frac{z}{6}+z^3\log(z)+z^2\log(2\pi)-2\log(\Gamma(z))z-2z\log(G(z))-2z\log(A)$

It's possible to obtain a more general formula for $\int_0^\infty \frac{\log\left(x^2+z^2\right)}{e^{a\pi x}-1}\sin\left(2\arctan\left(\frac{x}{z}\right)\right)\left(x^2+z^2\right)\mathrm{d}x$

Where a and z both positiv number.

$-\frac{3z^3}{2}+\frac{2z}{3a^2}+z^3\log\left(\frac{az}{2}\right)+\log(2)\left(2\frac{z^2}{a}+\frac{2z}{3a^2}\right)-\frac{2\log(a)z}{3a^2}+2\frac{z^2\log(\pi)}{a}-8\frac{\log\left(\Gamma\left(\frac{az}{2}\right)\right)z}{a^2}-8\frac{z\log\left(G\left(\frac{az}{2}\right)\right)}{a^2}-8\frac{z\log(A)}{a^2}$

6 The Melzak's product

We find the Melzak's product in the paper Multiple Gamma Function and Its Application to Computation of Series and Products page 19.

We see

$$\lim_{N\to\infty}\left(\prod_{k=1}^{2N}\left(1-4\frac{x^2}{k^2}\right)^{-k^2(-1)^k}\right)=\frac{\Gamma_3\left(\frac{3}{2}-x\right)^8\Gamma_3\left(\frac{3}{2}+x\right)^8\cos(\pi x)(G(1-x))^4(G(1+x))^4}{\Gamma_3(1-x)^8\Gamma_3(x+1)^8\pi}e^{2x^2+\frac{7\zeta(3)}{2\pi^2}}$$

There are a mistake about the power and the real formula is:

$$\lim_{N\to\infty}\left(\prod_{k=1}^{2N}\left(1-4\frac{x^2}{k^2}\right)^{-k^2(-1)^k}\right)=\frac{\Gamma_3\left(\frac{3}{2}-x\right)^{-8}\Gamma_3\left(\frac{3}{2}+x\right)^{-8}\cos(\pi x)(G(1-x))^4(G(1+x))^4}{\Gamma_3(1-x)^{-8}\Gamma_3(x+1)^{-8}\pi}e^{2x^2+\frac{7\zeta(3)}{2\pi^2}}$$

I see on a other paper the notation $G_3(z)$ but nevermind because $G_3(z)=\Gamma_3(z)$ and consequently we have the same mistake.

7 Applications

First example

Consider and calculate the closed form

$$\int_0^{\infty} \frac{x^2}{\mathrm{e}^{3\,\pi\,x}-1} \arctan\left(\frac{x}{2}\right) \mathrm{d}x$$

So we see a=3 and z=2

We obtain

$-\frac{64}{27}+\frac{2\,\log(\pi)}{3}+\frac{14\,\log(2)}{27}+\frac{4\,\log(3)}{3}-\frac{8\,\log(A)}{9}-\frac{4\,\zeta(1,-2)}{27}$

Or if you prefer, using the relationship $\zeta\left(1,-2\right)=-\frac{\zeta(3)}{4\,\pi^2}$

Where $\zeta\left(3\right)$ is the Apery's constant (10)

$-\frac{64}{27}+\frac{2\,\log(\pi)}{3}+\frac{14\,\log(2)}{27}+\frac{4\,\log(3)}{3}-\frac{8\,\log(A)}{9}+\frac{\zeta(3)}{27\,\pi^2}$

Second example

Consider and calculate the closed form

$$\int_0^{\infty} \frac{1}{\mathrm{e}^{4\,\pi\,x}-1} \arctan\left(\frac{3\,x}{2}\right) \cos\left(2\,\arctan\left(\frac{3\,x}{2}\right)\right) \left(x^2+\frac{4}{9}\right) \mathrm{d}x$$

So we see z=2/3 and a=4

We obtain

$\frac{473}{1944}+\frac{2\,\log(\Gamma(1/3))}{9}-\frac{\log(\pi)}{9}-\frac{32\,\log(2)}{81}-\frac{19\,\log(3)}{648}+\frac{\pi\,\sqrt{3}}{324}-\frac{\Psi\left(1,\frac{1}{3}\right)\sqrt{3}}{216\,\pi}-\frac{\log(A)}{18}+\frac{\zeta\left(1,-2,\frac{1}{3}\right)}{16}$

Where $\Psi\left(1,\frac{1}{3}\right)$ is the trigamma function at 1/3 (11)

I use the relation $\zeta\left(1,-2,1+t\right)=t^2\log\left(t\right)+\zeta\left(1,-2,t\right)$ with t=1/3

Third example

Consider and calculate the closed form

$$\int_0^{\infty} \frac{1}{e^{\frac{1}{2}\pi x} - 1} \log\left(x^2+9\right) \sin\left(2 \arctan\left(x/3\right)\right) \left(x^2+9\right) dx$$

So we see z=3 and a=1/2

We obtain

$$-\frac{83}{2} + 72 \log\left(\Gamma\left(1/4\right)\right) - 36 \log\left(\pi\right) - 38 \log\left(2\right) + 27 \log\left(3\right) - 24 \frac{K}{\pi} + 12 \log\left(A\right)$$

Fourth example

Consider and calculate the closed form $\lim_{N\to\infty} \left(\prod_{k=1}^{2N} \left(1 - \frac{9}{4k^2}\right)^{-k^2(-1)^k} \right)$

So we see x=3/4

$$\Gamma_3\left(\frac{3}{4}\right) = e^{-\frac{3K}{16\pi} - \frac{9}{128} + \frac{27 \log(A)}{32} - \frac{\zeta\left(1,-2,\frac{1}{4}\right)}{2} + \frac{19\zeta(3)}{128\pi^2} + \frac{5 \log(\pi)}{32} + \frac{5 \log(2)}{64} - \frac{5 \log(\Gamma(1/4))}{32}}$$

$$\Gamma_3\left(\frac{9}{4}\right) = e^{-\frac{3K}{16\pi} + \frac{9}{128} - \frac{27 \log(A)}{32} + \frac{\zeta\left(1,-2,\frac{1}{4}\right)}{2} + \frac{\zeta(3)}{8\pi^2} + \frac{5 \log(\Gamma(1/4))}{32}}$$

$$\Gamma_3\left(\frac{1}{4}\right) = e^{\frac{5K}{16\pi} - \frac{15}{128} + \frac{45 \log(A)}{32} + \frac{\zeta\left(1,-2,\frac{1}{4}\right)}{2} + \frac{\zeta(3)}{8\pi^2} + \frac{21 \log(\Gamma(1/4))}{32}}$$

$$\Gamma_3\left(\frac{7}{4}\right) = e^{-\frac{3 \log(2)}{64} + \frac{K}{16\pi} + \frac{3}{128} - \frac{9 \log(A)}{32} - \frac{\zeta\left(1,-2,\frac{1}{4}\right)}{2} + \frac{19\zeta(3)}{128\pi^2} - \frac{3 \log(\pi)}{32} + \frac{3 \log(\Gamma(1/4))}{32}}$$

$$\frac{\Gamma_3\left(\frac{3}{4}\right)^{-8} \Gamma_3\left(\frac{9}{4}\right)^{-8}}{\Gamma_3\left(\frac{1}{4}\right)^{-8} \Gamma_3\left(\frac{7}{4}\right)^{-8}} = \frac{\left(\Gamma\left(\frac{1}{4}\right)\right)^6}{2\pi^2} e^{-\frac{3}{4} + 6\frac{K}{\pi} + 9 \log(A)}$$

$$\left(G\left(\frac{1}{4}\right)\right)^4 \left(G\left(\frac{7}{4}\right)\right)^4 = \frac{\pi^3}{\left(\Gamma\left(\frac{1}{4}\right)\right)^6} 2^{\frac{3}{2}} e^{\frac{3}{4} - 9 \log(A)}$$

Finally we obtain

$$-e^{\frac{9}{8} + \frac{7\zeta(3)}{2\pi^2} + 6\frac{K}{\pi}}$$

8 References

(1): V. S. Adamchik, Contributions to the theory for the Barnes function (2003)

(2): V. S. Adamchik, Multiple Gamma Function and Its Application to Computation of Series and Products (2005)

(3): E. W. Barnes. The Theory of the G-function. Quart. J. Pure Appl. Math. 31, pages 264–314, 1899

(4),(5),(6) and (7): https://dlmf.nist.gov/5.17

(8): https://mathworld.wolfram.com/Glaisher-KinkelinConstant.html

(9): https://mathworld.wolfram.com/CatalansConstant.html

(10): https://mathworld.wolfram.com/AperysConstant.html

(11): https://mathworld.wolfram.com/TrigammaFunction.html