# ON THE SCHOLZ CONJECTURE

T. AGAMA

Abstract. In this paper we prove an inequality relating the length of addition chains producing number of the form $2^n - 1$ to the length of their shortest addition chain producing their exponents. In particular, we obtain the inequality

$$\delta(2^n - 1) \leq n - 1 + \iota(n) + G(n)$$

where $\delta(n)$ and $\iota(n)$ denotes the length of an addition chain and the shortest addition chain producing $n$, respectively, with $G : \mathbb{N} \longrightarrow \mathbb{R}$.

## 1. Introduction

An addition chain producing $n \geq 3$, roughly speaking, is a sequence of numbers of the form $1, 2, s_3, s_4, \ldots, s_{k-1}, s_k = n$ where each term is the sum of two earlier terms in the sequence, obtained by adding each sum generated to an earlier term in the sequence. The number of terms in the sequence excluding $n$ is the length of the chain. There are quite a number of addition chains producing a fixed number $n$. Among them the shortest is regarded as the shortest or optimal addition chain producing $n$. Nonetheless minimizing an addition chain can be an arduous endeavour, given that there are currently no efficient method for obtaining the shortest addition producing a given number. This makes the theory of addition chains an interesting subject to study. By letting $\iota(n)$ denotes the length of the shortest addition chain producing $n$, Arnold scholz conjectured the inequality

**Conjecture 1.1** (Scholz)**.** The inequality holds

$$\iota(2^n - 1) \leq n - 1 + \iota(n).$$

It has been shown computationally that the conjecture holds for all $n \leq 5784688$ and in fact it is an equality for all $n \leq 64$ [2]. Alfred Brauer proved the scholz conjecture for the star addition chain, an addition chain where each term obtained by summing uses the immediately subsequent number in the chain. By denoting the shortest length of the star addition chain by $\iota^*(n)$, it is shown that (See,[1])

**Theorem 1.1.** *The inequality holds*

$$\iota^*(2^n - 1) \leq n - 1 + \iota^*(n).$$

In this paper we study short addition chains producing numbers of the form $2^n - 1$ and the scholz conjecture. We obtain some crude and much more weaker inequalities related to the scholz conjecture.

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## 2. **Sub-addition chains**

In this section we introduce the notion of sub-addition chains.

**Definition 2.1.** Let $n \geq 3$, then by the addition chain of length $k-1$ producing $n$ we mean the sequence

$$1, 2, \ldots, s_{k-1}, s_k$$

where each term $s_j$ $(j \geq 3)$ in the sequence is the sum of two earlier terms, with the corresponding sequence of partition

$$2 = 1+1, \ldots, s_{k-1} = a_{k-1} + r_{k-1}, s_k = a_k + r_k = n$$

with $a_{i+1} = a_i + r_i$ and $a_{i+1} = s_i$ for $2 \leq i \leq k$. We call the partition $a_i + r_i$ the $i$ th **generator** of the chain for $2 \leq i \leq k$. We call $a_i$ the **determiners** and $r_i$ the **regulator** of the $i$ th generator of the chain. We call the sequence $(r_i)$ the regulators of the addition chain and $(a_i)$ the determiners of the chain for $2 \leq i \leq k$.

**Definition 2.2.** Let the sequence $1, 2, \ldots, s_{k-1}, s_k = n$ be an addition chain producing $n$ with the corresponding sequence of partition

$$2 = 1+1, \ldots, s_{k-1} = a_{k-1} + r_{k-1}, s_k = a_k + r_k = n.$$

Then we call the sub-sequence $(s_{j_m})$ for $1 \leq j \leq k$ and $1 \leq m \leq t \leq k$ a **sub-addition** chain of the addition chain producing $n$. We say it is **complete** sub-addition chain of the addition chain producing $n$ if it contains exactly the first $t$ terms of the addition chain. Otherwise we say it is an **incomplete** sub-addition chain.

## 3. **Addition chains of numbers of special forms**

In this section we study addition chains of numbers of special forms. We examine ways of minimizing the length of addition chains for numbers of the form $2^n - 1$.

**Lemma 3.1.** *Let $\iota(n)$ denotes the shortest length of an addition chain producing $n$. Then the lower bound holds*

$$\iota(n) > \frac{\log n}{\log 2} - 1.$$

*Remark* 3.2. We now obtain an inequality related to scholz conjecture.

**Theorem 3.3.** *Let $\delta(n)$ and $\iota(n)$ denotes the length of an addition chain and the shortest addition chain producing n, respectively. Then there exists some $G : \mathbb{N} \longrightarrow \mathbb{R}$ such that*

$$\delta(2^n - 1) \leq n - 1 + \iota(n) + G(n).$$

*Proof.* First, let us construct the shortest addition chain producing $2^n$ as $1, 2, 2^2, \ldots,$ $2^{n-1}, 2^n$ with corresponding sequence of partition

$$2 = 1+1, 2+2 = 2^2, 2^2 + 2^2 = 2^3 \ldots, 2^{n-1} = 2^{n-2} + 2^{n-2}, 2^n = 2^{n-1} + 2^{n-1}$$

with $a_i = 2^{i-2} = r_i$ for $2 \leq i \leq n+1$, where $a_i$ and $r_i$ denotes the determiner and the regulator of the $i$ th generator of the chain. Let us consider only the complete sub-addition chain

$$2 = 1+1, 2+2 = 2^2, 2^2+2^2 = 2^3 \ldots, 2^{n-1} = 2^{n-2} + 2^{n-2}.$$

Next we extend this complete sub-addition chain by adjoining the sequence

$$2^{n-1} + 2^{\lfloor \frac{n-1}{2} \rfloor}, 2^{n-1} + 2^{\lfloor \frac{n-1}{2} \rfloor} + 2^{\lfloor \frac{n-1}{2^2} \rfloor} \ldots, 2^{n-1} + 2^{\lfloor \frac{n-1}{2} \rfloor} + 2^{\lfloor \frac{n-1}{2^2} \rfloor} + \cdots + 2^1.$$

We note that the adjoined sequence contributes at most

$$\lfloor \frac{\log n}{\log 2} \rfloor \leq \iota(n)$$

terms to the original complete sub-addition chain, where the upper bound follows from Lemma 3.1. Since the inequality holds

$$\begin{aligned} 2^{\lfloor \frac{n-1}{2} \rfloor} + 2^{\lfloor \frac{n-1}{2^2} \rfloor} + \cdots + 2^1 &< \sum_{j=0}^{n-1} 2^j \\ &= 2^n - 1 \end{aligned}$$

there exists some $G : \mathbb{N} \longrightarrow \mathbb{R}$ counting the terms in the remaining terms of the addition chain producing $2^n - 1$. This completes the proof. □

Scholz's conjecture is the claim

**Conjecture 3.1** (Scholz)**.** The inequality holds

$$\iota(2^n - 1) \leq n - 1 + \iota(n).$$

This result would follow from the inequality established if we can take $\delta(2^n - 1) = \iota(2^n - 1)$ and choose $G(n) = 0$ in Theorem 3.3. In other words, the scholz conjecture can be reformulated in the following way:

**Conjecture 3.2** (Scholz)**.** If $\delta(2^n - 1) = \iota(2^n - 1)$, then the inequality holds

$$\delta(2^n - 1) \leq n - 1 + \iota(n) + G(n)$$

with $G(n) = 0$ for all $n \in \mathbb{N}$.

[1].

## References

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Department of Mathematics, African Institute for Mathematical science, Ghana
*E-mail address*: `theophilus@aims.edu.gh/emperordagama@yahoo.com`

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