A Method to Prove a Prime Number between 3*N* and 4*N*

Wing K. Yu

Abstract

In this paper, we will prove that when an integer n >1, there exists a prime number between 3n and 4n. This is another step in the expansion of the Bertrand's postulate / Chebyshev's theorem after the proof of a prime number between 2n and 3n.

Introduction

The Bertrand's postulate / Chebyshev's theorem States that for any positive integer n, there is always a prime number p such that $n < p \leq 2n$. It was proved by Pafnuty Chebyshev in 1850 [1]. In 2006, M. El Bachraoui [2] expanded the theorem by proving that for any positive integer n, there is a prime number p such that $2n < p \leq 3n$. In 2011, Andy Loo [3] expanded the theorem to that when $n \geq 2$, there exists a prime number in the interval $(3n, 4n)$. Recently, the author used a different method to prove that a prime number exists between 2n and 3n by analyzing the binomial coefficient $\binom{3n}{n}$. In this paper, we will use the similar way to prove that a prime number exists between 3n and 4n by analyzing the binomial coefficient $\binom{4n}{n}$. We will cite some important concepts from the previous paper [4].

Definition: $\Gamma_{a\geq p\geq b}\{n\}$ denotes the prime number decomposition operator. It is the product of the prime numbers in the decomposition of a positive integer n or a positive integer expression. In this operator, p is a prime number, a and b are real numbers, and $n \geq a \geq p > b \geq 1$.

It has some properties: It is always true that $\Gamma_{a\geq p\geq b}\{n\} \geq 1$ **— (1)**

If no prime number in $\Gamma_{a\geq p\geq b}\{n\}$, then $\Gamma_{a\geq p\geq b}\{n\}$ = 1, or vice versa, if $\Gamma_{a\geq p\geq b}\{n\}$ = 1, then no prime number in $\Gamma_{a\geq p\geq b}\{n\}$ as in $\Gamma_{12\geq p\geq 4}\{12\} = 11^0 \cdot 7^0 \cdot 5^0 = 1$. **— (2)**

If there is at least one prime number in $\Gamma_{a\geq p\geq b}\{n\}$, then $\Gamma_{a\geq p\geq b}\{n\} > 1$, or vice versa, if $\Gamma_{a\geq p\geq b}\{n\} > 1$, then there is at least one prime number in $\Gamma_{a\geq p\geq b}\{n\}$. **— (3)**

We define *R(p)* by the inequalities $p^{R(p)} \leq 4n < p^{R(p)+1}$, and determine the p-$adic\ valuation$ of $\binom{4n}{n}$. $v_p\left(\binom{4n}{n}\right) = v_p((4n)!) - v_p((3n)!) - v_p(n!) = \sum_{i=1}^{R(p)}\left(\left\lfloor\frac{4n}{p^i}\right\rfloor - \left\lfloor\frac{3n}{p^i}\right\rfloor - \left\lfloor\frac{n}{p^i}\right\rfloor\right) \leq R(p)$ because for any real numbers a and b, the expression of $\lfloor a+b\rfloor - \lfloor a\rfloor - \lfloor b\rfloor$ is 0 or 1.

Thus, if p divides $\binom{4n}{n}$, then $v_p\left(\binom{4n}{n}\right) \le R(p) \le \log_p(4n)$, or $p^{v_p\left(\binom{3n}{n}\right)} \le p^{R(p)} \le 4n$ **— (4)**

And if $4n \ge p > \lfloor 2\sqrt{n} \rfloor$, then $0 \le v_p\left(\binom{4n}{n}\right) \le R(p) \le 1$. **— (5)**

From the prime number decomposition,

$$\binom{4n}{n} = \frac{(4n)!}{n!\cdot(3n)!} = \Gamma_{4n\ge p>n}\left\{\frac{(4n)!}{n!\cdot(3n)!}\right\} \cdot \Gamma_{n\ge p>\lfloor 2\sqrt{n}\rfloor}\left\{\frac{(4n)!}{n!\cdot(3n)!}\right\} \cdot \Gamma_{\lfloor 2\sqrt{n}\rfloor \ge p}\left\{\frac{(4n)!}{n!\cdot(3n)!}\right\}.$$

Since all prime numbers in $n!$ are not in the range of $4n \ge p > n$,

$$\Gamma_{4n\ge p>n}\left\{\frac{(4n)!}{n!\cdot(3n)!}\right\} = \Gamma_{4n\ge p>n}\left\{\frac{(4n)!}{(3n)!}\right\}.$$

Referring to **(5)**, $\Gamma_{n\ge p>\lfloor 2\sqrt{n}\rfloor}\left\{\frac{(4n)!}{n!\cdot(3n)!}\right\} \le \prod_{n\ge p} p$.

It has been proved [5] that $\prod_{n\ge p} p < 2^{2n-3}$ when $n \ge 3$.

Thus for $n \ge 3$, $\binom{4n}{n} < \Gamma_{4n\ge p>n}\left\{\frac{(4n)!}{(3n)!}\right\} \cdot 2^{2n-3} \cdot \Gamma_{\lfloor 2\sqrt{n}\rfloor \ge p}\left\{\frac{(4n)!}{n!\cdot(3n)!}\right\}$ **— (6)**

Proposition

For every integer $n > 1$, there exists at least a prime number p such that $3n < p \le 4n$.

Proof:

By induction on n, for n=2, $\binom{4n}{n} = \binom{8}{2} = 28 > \frac{4^{4n-3}}{n\cdot 3^{3n-3}} = \frac{512}{27} \approx 18.96$

If $\binom{4n}{n} > \frac{4^{4n-3}}{n\cdot 3^{3n-3}}$ for n stands, then for n+1,

$$\binom{4(n+1)}{(n+1)} = \frac{(4n+4)(4n+3)(4n+2)(4n+1)}{(n+1)(3n+3)(3n+2)(3n+1)} \cdot \binom{4n}{n}$$

$$> \frac{(4n+4)(4n+3)(4n+2)(4n+1)}{(n+1)(3n+3)(3n+2)(3n+1)} \cdot \frac{4^{4n-3}}{n\cdot 3^{3n-3}} = \frac{4}{3} \cdot \frac{4n+3}{3n+2} \cdot \frac{4n+2}{3n+1} \cdot \frac{4n+1}{n} \cdot \frac{4^{4n-3}}{(n+1)\cdot 3^{3n-3}}$$

$$> \frac{4}{3} \cdot \frac{4}{3} \cdot \frac{4}{3} \cdot \frac{4}{1} \cdot \frac{4^{4n-3}}{(n+1)\cdot 3^{3n-3}} = \frac{4^{4(n+1)-3}}{(n+1)\cdot 3^{3(n+1)-3}}$$

Thus for $n \ge 2$, $\binom{4n}{n} > \frac{4^{4n-3}}{n\cdot 3^{3n-3}}$ **— (7)**

Applying **(7)** into **(6)**:

For $n \ge 3$, $\frac{4^{4n-3}}{n\cdot 3^{3n-3}} < \binom{4n}{n} < \Gamma_{4n\ge p>n}\left\{\frac{(4n)!}{(3n)!}\right\} \cdot 2^{2n-3} \cdot \Gamma_{\lfloor 2\sqrt{n}\rfloor \ge p}\left\{\frac{(4n)!}{n!\cdot(3n)!}\right\}$ **— (8)**

Let *π(x)* be the number of prime numbers less than or equal to x, where x is a positive real number. For the first six sequential natural numbers, there are three prime numbers 2, 3, and 5. For adding any successive set of six sequential natural numbers, there are at most two prime numbers added, $p \equiv 1$ (MOD 6) and $p \equiv 5$ (MOD 6). Thus, $\pi(x) \le \left\lfloor \frac{x}{3} \right\rfloor + 2 \le \frac{x}{3} + 2$. **— (9)**

Referring to **(4)** and **(9)**,

$$\Gamma_{\lfloor 2\sqrt{n}\rfloor \geq p}\left\{\frac{(4n)!}{n!\cdot(3n)!}\right\} = \Gamma_{\lfloor 2\sqrt{n}\rfloor \geq p}\left\{\binom{4n}{n}\right\} \leq (4n)^{\pi(2\sqrt{n})} \leq (4n)^{\frac{2\sqrt{n}}{3}+2} \qquad \textbf{— (10)}$$

Applying **(10)** into **(8)**: $\frac{4^{4n-3}}{n(3^{3n-3})} < \Gamma_{4n\geq p>n}\left\{\frac{(4n)!}{(3n)!}\right\} \cdot 2^{2n-3} \cdot (4n)^{\frac{2\sqrt{n}}{3}+2}$

Since both $2^{2n-3} > 0$ and $(4n)^{\frac{2\sqrt{n}}{3}+2} > 0$ for $n \geq 3$,

$$\Gamma_{4n\geq p>n}\left\{\frac{(4n)!}{(3n)!}\right\} > \frac{4^{4n-3}}{n(3^{3n-3})(2^{2n-3})(4n)^{\frac{2\sqrt{n}}{3}+2}} = \frac{27\cdot\left(\frac{4}{3}\right)^{3n}}{2\cdot(4n)^{\frac{2\sqrt{n}}{3}+3}} = \frac{27}{2}\cdot\frac{\left(\frac{4}{3}\right)^{3n}}{(4n)^{\frac{2\sqrt{n}+9}{3}}} \qquad \textbf{— (11)}$$

Let $f(x) = \frac{u}{w}$ where x, u, w are real numbers and $x \geq 42$, $u = \frac{27}{2}\cdot\left(\frac{4}{3}\right)^{3x}$, $w = (4x)^{\frac{2\sqrt{x}+9}{3}}$

$$\frac{du}{dx} = \left(\frac{27}{2}\cdot\left(\frac{4}{3}\right)^{3x}\right)' = \frac{27}{2}\cdot\left(\frac{4}{3}\right)^{3x}\cdot 3\cdot ln\left(\frac{4}{3}\right) = u\cdot 3\cdot ln\left(\frac{4}{3}\right)$$

$$\frac{dw}{dx} = \left((4x)^{\frac{2\sqrt{x}+9}{3}}\right)' = \left((4x)^{\frac{2\sqrt{x}+9}{3}}\right)\left(\frac{ln(4x)}{3\sqrt{x}} + \frac{2\sqrt{x}+9}{3x}\right) = w\left(\frac{ln(x)+ln(4)+2}{3\sqrt{x}} + \frac{3}{x}\right)$$

$$f'(x) = \left(\frac{u}{w}\right)' = \frac{w(u)' - u(w)'}{w^2} = \frac{u}{w}\left(3\cdot ln\left(\frac{4}{3}\right) - \frac{ln(x)+ln(4)+2}{3\sqrt{x}} - \frac{3}{x}\right)$$

Let $f_1(x) = 3\cdot ln\left(\frac{4}{3}\right) - \frac{ln(x)+ln(4)+2}{3\sqrt{x}} - \frac{3}{x}$

Since $f_1{}'(x) = \frac{ln(x)+ln(4)}{6x\sqrt{x}} + \frac{3}{x^2} > 0$, when $x > 1$, $f_1(x)$ is a strictly increasing function.

When $x = 42$, $f_1(x) = 3\cdot ln\left(\frac{4}{3}\right) - \frac{ln(x)+ln(4)+2}{3\sqrt{x}} - \frac{3}{x} \approx 0.863 - 0.367 - 0.071 = 0.425 > 0$.

Thus, when $x \geq 42$, $f_1(x) > 0$.

Since when $x \geq 42$, u, w, and $f_1(x)$ are greater than zero, $f'(x) = \frac{u}{w}\cdot f_1(x) > 0$.

Thus $f(x)$ is a strictly increasing function for $x \geq 42$. Then when $x \geq 42$, $f(x+1) > f(x)$.

Let $n = \lfloor x\rfloor \geq 42$, then $f(n+1) > f(n) = \frac{27}{2}\cdot\frac{\left(\frac{4}{3}\right)^{3n}}{(4n)^{\frac{2\sqrt{n}+9}{3}}}$

Since for $n = 42$, $f(n) = \frac{27}{2}\cdot\frac{\left(\frac{4}{3}\right)^{3n}}{(4n)^{\frac{2\sqrt{n}+9}{3}}} = \frac{27}{2}\cdot\frac{\left(\frac{4}{3}\right)^{126}}{(168)^{\frac{2\sqrt{42}+9}{3}}} \approx \frac{7.457E+16}{1.952E+16} > 1$, and since

$f(n+1) > f(n)$, by induction on n, when $n \geq 42$, $f(n) = \frac{27}{2}\cdot\frac{\left(\frac{4}{3}\right)^{3n}}{(4n)^{\frac{2\sqrt{n}+9}{3}}} > 1$. **— (12)**

Applying **(12)** to **(11)**: When $n \geq 42$, $\Gamma_{4n\geq p>n}\left\{\frac{(4n)!}{(3n)!}\right\} > \frac{27}{2}\cdot\frac{\left(\frac{4}{3}\right)^{3n}}{(4n)^{\frac{2\sqrt{n}+9}{3}}} > 1$.

Thus when $n \geq 42$,

$\Gamma_{4n \geq p > n}\{\frac{(4n)!}{(3n)!}\}$

= $\Gamma_{4n \geq p > 3n}\{\frac{(4n)!}{(3n)!}\} \cdot \Gamma_{3n \geq p > 2n}\{\frac{(4n)!}{(3n)!}\} \cdot \Gamma_{2n \geq p > \frac{3n}{2}}\{\frac{(4n)!}{(3n)!}\} \cdot \Gamma_{\frac{3n}{2} \geq p > \frac{4n}{3}}\{\frac{(4n)!}{(3n)!}\} \cdot \Gamma_{\frac{4n}{3} \geq p > n}\{\frac{(4n)!}{(3n)!}\} > 1$.

When $3n \geq p > 2n$ in $\left(\frac{(4n)!}{(3n)!}\right)$, if $v_p((4n)!)$ has a factor of p then $v_p((3n)!)$ also has a factor of p.

Thus, when $3n \geq p > 2n$, $v_p\left(\frac{(4n)!}{(3n)!}\right) = v_p((4n)!) - v_p((3n)!) = 1 - 1 = 0$.

Since p^0=1, referring to **(2)**, $\Gamma_{3n \geq p > 2n}\{\frac{(4n)!}{(3n)!}\} = 1$.

When $\frac{3n}{2} \geq p > \frac{4n}{3}$ in $\left(\frac{(4n)!}{(3n)!}\right)$, if $v_p((4n)!)$ has a factor of p, it is p^2 in $p \cdot 2p$, then $v_p((3n)!)$ also has a factor of p^2 in $p \cdot 2p$. Thus, $v_p\left(\frac{(4n)!}{(3n)!}\right) = v_p((4n)!) - v_p((3n)!) = 2 - 2 = 0$.

Since p^0=1, referring to **(2)**, $\Gamma_{\frac{3n}{2} \geq p > \frac{4n}{3}}\{\frac{(4n)!}{(3n)!}\} = 1$.

Thus, $\Gamma_{4n \geq p > n}\{\frac{(4n)!}{(3n)!}\} = \Gamma_{4n \geq p > 3n}\{\frac{(4n)!}{(3n)!}\} \cdot \Gamma_{2n \geq p > \frac{3n}{2}}\{\frac{(4n)!}{(3n)!}\} \cdot \Gamma_{\frac{4n}{3} \geq p > n}\{\frac{(4n)!}{(3n)!}\} > 1$ **— (13)**

Referring to **(1)**, $\Gamma_{4n \geq p > 3n}\{\frac{(4n)!}{(3n)!}\} \geq 1$, $\Gamma_{2n \geq p > \frac{3n}{2}}\{\frac{(4n)!}{(3n)!}\} \geq 1$, and $\Gamma_{\frac{4n}{3} \geq p > n}\{\frac{(4n)!}{(3n)!}\} \geq 1$.

If $n \geq 42$ and $\Gamma_{4n \geq p > 3n}\{\frac{(4n)!}{(3n)!}\} > 1$, then referring to **(3)**, there exists at least a prime number p such that $3n < p \leq 4n$. **— (14)**

$\Gamma_{2n \geq p > \frac{3n}{2}}\{\frac{(4n)!}{(3n)!}\} = \Gamma_{4 \cdot (\frac{n}{2}) \geq p > 3 \cdot (\frac{n}{2})}\{\frac{(4n)!}{(3n)!}\}$.

If $\frac{n}{2} \geq 21$ and, $\Gamma_{4 \cdot (\frac{n}{2}) \geq p > 3 \cdot (\frac{n}{2})}\{\frac{(4n)!}{(3n)!}\} > 1$, let $m_1 = \frac{n}{2}$, then when $m_1 \geq 21$, there exists at least a prime number p such that $3m_1 < p \leq 4m_1$. Since $n \geq 42 > m_1 \geq 21$, the statement is also valid for n. Thus, when $n \geq 42$, then $\Gamma_{4n \geq p > 3n}\{\frac{(4n)!}{(3n)!}\} > 1$ **— (15)**

$\Gamma_{\frac{4n}{3} \geq p > n}\{\frac{(4n)!}{(3n)!}\} = \Gamma_{4 \cdot (\frac{n}{3}) \geq p > 3 \cdot (\frac{n}{3})}\{\frac{(4n)!}{(3n)!}\}$.

If $\frac{n}{3} \geq 14$ and, $\Gamma_{4 \cdot (\frac{n}{3}) \geq p > 3 \cdot (\frac{n}{3})}\{\frac{(4n)!}{(3n)!}\} > 1$, let $m_2 = \frac{n}{3}$, then when $m_2 \geq 14$, there exists at least a prime number p such that $3m_2 < p \leq 4m_2$. Since $n \geq 42 > m_2 \geq 14$, the statement is also valid for n. Thus, when $n \geq 42$, then $\Gamma_{4n \geq p > 3n}\{\frac{(4n)!}{(3n)!}\} > 1$ **— (16)**

From the right side of **(13)**, at least one of these 3 factors is greater than one when $n \geq 42$. From **(14)**, **(15)**, and **(16)**, when $n \geq 42$ and any one of these 3 factors is greater than one, there exists at least a prime number p such that $3n < p \leq 4n$. **— (17)**

Table 1 shows that when $2 \le n \le 42$, there is a prime number p such that $3n < p \le 4n$. **— (18)**

Thus, the proposition is proven by combining **(17)** and **(18)**: For every integer $n>1$, there exists at least a prime number p such that $3n < p \le 4n$. **— (19)**

Table 1: For $2 \le n \le 42$, there is a prime number p such that $3n < p \le 4n$.

$3n$	6	9	12	15	18	21	24	27	30	33	36	39	42	45
p	7	11	13	17	19	23	29	31	37	41	43	47	53	59
$4n$	8	12	16	20	24	28	32	36	40	44	48	52	56	60
$3n$	48	51	54	57	60	63	66	69	72	75	78	81	84	87
p	61	67	71	73	79	83	83	89	89	97	97	101	101	103
$4n$	64	68	72	76	80	84	88	92	96	100	104	108	112	116
$3n$	90	93	96	99	102	105	108	111	114	117	120	123	126	
p	103	107	107	109	109	113	113	127	127	131	131	137	139	
$4n$	120	124	128	132	136	140	144	148	152	156	160	164	168	

References

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[5] Wikipedia, Lemma 4, https://en.wikipedia.org/wiki/Proof_of_Bertrand%27s_postulate