A Proof that $\zeta(n \geq 2)$ is Irrational

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Abstract

We show that using the denominators of the terms of $\zeta(n)-1 = z_n$ as decimal bases gives all rational numbers in (0,1) as single decimals. We also show the partial sums of z_n are not given by such single digits using the partial sum's terms. These two properties yield a proof that z_n is irrational.

1 Introduction

Apery's $\zeta(3)$ is irrational proof [1] and its simplifications [3, 8] are the only proofs that a specific odd argument for $\zeta(n)$ is irrational. The irrationality of even arguments of zeta are a natural consequence of Euler's formula [2]:

$$\zeta(2n) = \sum_{k=1}^{\infty} \frac{1}{k^{2n}} = (-1)^{n-1} \frac{2^{2n-1}}{(2n!)} B_{2n} \pi^{2n}. \tag{1}$$

Apery also showed $\zeta(2)$ is irrational, and Beukers, based on the work (tangentially) of Apery, simplified both proofs. He replaced Apery's mysterious recursive relationships with multiple integrals. See Poorten [9] for the history of Apery's proof; Havil [5] gives an overview of Apery's ideas and attempts to demystify them. Also of interest is Huylebrouck's [6] paper giving an historical context for the main technique used by Beukers.

Attempts to generalize the techniques of the one odd success seem to be hopelessly elusive. Apery's and other ideas can be seen in the work of Rivoal and Zudilin [10, 11]. Their results, that there are an infinite number of odd n such that $\zeta(n)$ is irrational and at least one of the cases 5,7,9, 11 likewise irrational do suggest a radically different approach is necessary.

Let

$$z_n = \zeta(n) - 1 = \sum_{j=2}^{\infty} \frac{1}{j^n} \text{ and } s_k^n = \sum_{j=2}^{k} \frac{1}{j^n}.$$

We show that every rational number in $(0, 1)$ can be written as a single decimal using the denominators of a term in z_n as a number basis. But the partial sums can't be expressed with such a single decimal. These two properties yield a proof that all z_n are irrational.

Properties of z_n

We define a decimal set.

Definition 1. *Let*

$$d_{j^n} = \{1/j^n, \ldots, (j^n - 1)/j^n\} = \{.1, \ldots, .(j^n - 1)\} \textit{ base } j^n.$$

That is d_{j^n} consists of all single decimals greater than 0 and less than 1 in base j^n. The decimal set for j^n is

$$D_{j^n} = d_{j^n} \setminus \bigcup_{k=2}^{j-1} d_{k^n}.$$

The set subtraction removes duplicate values.

Definition 2.

$$\bigcup_{j=2}^{k} D_{j^n} = \Xi_k^n$$

The union of decimal sets gives all rational numbers in $(0, 1)$.

Lemma 1.

$$\bigcup_{j=2}^{\infty} D_{j^n} = \mathbb{Q}(0, 1)$$

Proof. Every rational $a/b \in (0, 1)$ is included in a d_{b^n} and hence in some D_{r^n} with $r \leq b$. This follows as $ab^{n-1}/b^n = a/b$ and as $a < b$, per $a/b \in (0, 1)$, $ab^{n-1} < b^n$ and so $a/b \in d_{b^n}$. □

Next we show $s_k^n \notin \Xi_k^n$.

Lemma 2. *If $s_k^n = r/s$ with r/s a reduced fraction, then 2^n divides s.*

Proof. The set $\{2, 3, \ldots, k\}$ will have a greatest power of 2 in it, a; the set $\{2^n, 3^n, \ldots, k^n\}$ will have a greatest power of 2, na. Also $k!$ will have a powers of 2 divisor with exponent b; and $(k!)^n$ will have a greatest power of 2 exponent of nb. Consider

$$\frac{(k!)^n}{(k!)^n} \sum_{j=2}^{k} \frac{1}{j^n} = \frac{(k!)^n/2^n + (k!)^n/3^n + \cdots + (k!)^n/k^n}{(k!)^n}. \tag{2}$$

The term $(k!)^n/2^{na}$ will pull out the most 2 powers of any term, leaving a term with an exponent of $nb - na$ for 2. As all other terms but this term will have more than an exponent of 2^{nb-na} in their prime factorization, we have the numerator of (2) has the form

$$2^{nb-na}(2A + B),$$

where $2 \nmid B$ and A is some positive integer. This follows as all the terms in the factored numerator have powers of 2 in them except the factored term $(k!)^n/2^{na}$. The denominator, meanwhile, has the factored form

$$2^{nb}C,$$

where $2 \nmid C$. This leaves 2^{na} as a factor in the denominator with no powers of 2 in the numerator, as needed. □

Lemma 3. *If $s_k^n = r/s$ with r/s a reduced fraction and p is a prime such that $k > p > k/2$, then p^n divides s.*

Proof. First note that $(k, p) = 1$. If $p|k$ then there would have to exist r such that $rp = k$, but by $k > p > k/2$, $2p > k$ making the existence of such a natural number $r > 1$ impossible.

The reasoning is much the same as in Lemma 2; cf. Chapter 2, Problem 21 in [2], solution in [7]. Consider

$$\frac{(k!)^n}{(k!)^n} \sum_{j=2}^{k} \frac{1}{j^n} = \frac{(k!)^n/2^n + \cdots + (k!)^n/p^n + \cdots + (k!)^n/k^n}{(k!)^n}. \tag{3}$$

As $(k, p) = 1$, only the term $(k!)^n/p^n$ will not have p in it. The sum of all such terms will not be divisible by p, otherwise p would divide $(k!)^n/p^n$. As $p < k$, p^n divides $(k!)^n$, the denominator of r/s, as needed. □

Lemma 4. *For any $k \geq 2$, there exists a prime p such that $k < p < 2k$.*

Proof. This is Bertrand's postulate [4]. $\square$

Theorem 1. *If $s_k^n = \frac{r}{s}$, with r/s reduced, then $s > k^n$.*

Proof. Using Lemma 4, for even k, we are assured that there exists a prime p such that $k > p > k/2$. If k is odd, $k-1$ is even and we are assured of the existence of prime p such that $k-1 > p > (k-1)/2$. As $k-1$ is even, $p \neq k-1$ and $p > (k-1)/2$ assures us that $2p > k$, as $2p = k$ implies k is even, a contradiction.

For both odd and even k, using Lemma 4, we have assurance of the existence of a p that satisfies Lemma 3. Using Lemmas 2 and 3, we have $2^n p^n$ divides the denominator of r/s and as $2^n p^n > k^n$, the proof is completed. $\square$

Corollary 1.

$$s_k^n \notin \Xi_k^n$$

Proof. This is a restatement of Theorem 1. $\square$

z_n is irrational

Theorem 2. *z_n is irrational.*

Proof. Suppose z_n is rational. Then, using Lemma 1, $z_n \in \Xi_k^n$ for some first k. Using Corollary 1 and convergence there exists a K such that for all $j > K$

$$z_n - s_j^n < \Xi_k^n - s_k^n = \epsilon_k. \tag{4}$$

It also follows that for all $j \leq K$

$$z_n - s_j^n > \epsilon_k. \tag{5}$$

These properties follow as s_j^n goes monotonically towards z_n. Also, we know a convergent point will be less than all partial sums, so the right hand side of (4) does not require an absolute value.

There are three possibilities for the relationship between the smallest j value and k: $k = j$, $k > j$, or $k < j$. We will show each gives a contradiction.

If $k = j$, noting (4) implies

$$\sum_{r=j+1}^{\infty} \frac{1}{r^n} = z_n - s_j^n < \Xi_k^n - s_k^n = \epsilon_k$$

and so adding s_k^n gives

$$\sum_{r=j+1}^{\infty} \frac{1}{r^n} + s_j^n = z_n < \Xi_k^n = \epsilon_k + s_k^n,$$

which implies $z_n < z_n$, a contradiction.

If $k > j$, then once again adding s_k^n gives

$$\sum_{r=j+1}^{k} \frac{1}{r^n} + \sum_{r=k+1}^{\infty} \frac{1}{r^n} + \sum_{r=2}^{k} \frac{1}{r^n} < \Xi_k^n,$$

but the left hand side of this inequality is greater than z_n and once again this implies $z_n + \delta < z_n$, where $\delta > 0$, a contradiction.

For $k < j$, as $k < K$, (5) gives $z_n - s_k^n > \epsilon_k$, whereas (4) stipulates $z_n - s_k^n = \epsilon_k$, another contradiction.

Our assumption of the rationality of z_n leads to a contradiction in all cases: z_n is irrational. $\square$

2 Conclusion

The property $s_k^n \notin \Xi_k^n$ means that the decimal bases required to represent partial sums are always greater than the bases given by the denominators of the partial's terms used as bases. As these latter bases encompass all candidate rational convergence points, partials can only get close to (have perfect approximations in single decimals) using ever changing and ever growing decimal bases. Whereas if the convergent point were rational these partials would get ever closer to a single decimal in a fixed decimal basis.

A source of confusion can be that decimal representations of any convergent series in a given base will have partials with an ever greater number of decimal digits. So $.\overline{1}$ repeating in base 10 will have partials with an ever greater number of digits, yet those points must converge to a fixed, single decimal in some base: .1, base 9 for a convergence point of 1/9. If this isn't true for a set of bases that includes all rational plausible convergence points as single digits, then the series must converge to an irrational number.

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