Sums of Biquadratics with integer coefficents

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Abstract

We consider two types of equations shown below:

$$ax^4 + by^4 = cz^4 + dw^4 \quad \text{---------------} \ (1)$$

$$ax^4 + by^4 + cz^4 = dw^4 \quad \text{---------------} \ (2)$$

Condition: product (abcd) not equal to zero

Existence of solution for Diophantine equation:

$ax^4 + by^4 = cz^4 + dw^4$ & $ax^4 + by^4 + cz^4 = dw^4$, are known if (abcd) is square number& product not equal to zero. So we are curious about whether above equation has a solution if (abcd) is not square number & product not equal to zero. In particular, when does this equation have infinitely many integer solutions? Bremner, A., & Choudhry, A., & Ulas [1] have showed the solution family of the similar equation ($ax^4 + by^4 + cz^4 + dw^4 = 0$) with infinitely many rational points using elliptic curve theory. We show other family of solution of this equation with infinitely many rational points. As a bonus we have considered an interesting case of equation ($ax^4 + by^4 = az^4 + bw^4$) in which the product of the co-eficents is a square number, but has been parameterized by using only algebraic methods without taking re-course to elliptic curve theory.

We have considered the below family of six equations:

(A) $$p(X^4 + Y^4) = 2Z^4 + W^4$$

(B) $$d^2(X)^4 + (d-2c)^2Y^4 = 2d(d-2c)Z^4 + 4W^4$$

(C) $$(d^2)(a^2)(X^4) + (-d+2a)^2(a^2)(Y^4) + 2d(-d+2a)Z^4 = 4(a^2)W^4$$

(D) $$pr^2(X)^4 + 4p(p+r)^2Y^4 + r^3(Z^4) = r^2(p+r)W\text{^}4$$

(E) $$a(p)^4 + b(q)^4 = c(r)^4 + b(d)^4$$

(F) $$(-c^2+4ec-2e^2)X^4 + (-c^2+4ec-2e^2)Y^4 + (2c-2e)^2Z^4 = 2W^4$$

Case (A):

$p(X^4 + Y^4) = 2Z^4 + W^4 - - - - - - - - - (1)$

We use an identity,

$$p(t+1)^4 + p(t)^4 = 2(t^2+at+b)^2 + (ct^2+dt+e)^2,$$

$$with,$$

$$p = 3 - 2d + \frac{(d)^2}{2},$$

$$e = 1, c = (-2+d),$$

$$a = (-d+3),$$

$$b = 1/2(2-d)$$

So, we look for the integer solutions,

$$\left\{Z^2 = t^2 + (-d+3)t + 1 - \frac{d}{2}, \quad W^2 = (-2+d)t^2 + dt + 1\right\}$$

By parameterizing the second equation and substituting the result to first equation, then we obtain quartic equation below.

$$u^2 = (-2d+4)k^4 + (-24+8d)k^3 + (32-4d)k^2 + (-8d^2-48+24d)k + (16+2d^3-4d^2)$$

This quartic equation has infinitely many rational solutions for

$$d = 6,8,18,24,26,42,48,50 \; with \; d <= 50.$$

$(d,p) = (6,9),(8,19),(18,129),(24,243),(26,289),(42,801),(48,1059),(50,1153).$

Hence we can obtain infinitely many integer solutions for equation (1) where,

$$p = (9,19,129,243,289,801) with\ p < 1000$$

For (d,p)=(6,9) we get:

$$9x^4 + 9y^4 = 2z^4 + w^4$$

$$(x,y,z,w) = (41,153,223,58)$$

<u>Case(B):</u>

$$d^2a^2X^4 + (-d+2a)^2a^2Y^4 + 2d(-d+2a)Z^4 = 4a^2W^4$$

$$d^2a^2X^4 + (-d+2a)^2a^2Y^4 + 2d(-d+2a)Z^4 = 4a^2W^4 --- (3)$$

(a,d) are arbitrary.

We use an identity:

$$p(t+1)^4 + q(t)^4 + r(at^2+at)^2 = s(at^2+dt+e)^2,$$

with $(p,q,r,s,e) =$

$$\left[\frac{1}{4}(sd^2), \frac{1}{4}(s(2a-d)^2), \frac{1}{2}(a^2)\big(sd(2a-d)\big), s, \frac{1}{2(d)}\right]$$

So, we look for the integer solutions:

$$\left\{Z^2 = at^2 + at, \quad W^2 = at^2 + dt + \frac{1}{2d}\right\}$$

By parameterizing the second equation and substituting the result to first equation, then we obtain quartic equation below.

$$u^2 = (4a-2d)k^4 + 2da^2$$

This quartic equation has infinitely many rational solutions for (a,d) with (a,d)<=5 below.

$$(a,d) = (1,3),(1,4),(1,5),(3,2),(3,4),(4,1),(4,2),(4,3),(4,4),(4,5),(5,2).$$

Hence we can obtain infinitely many integer solutions for equation (3).

For (a,d)=(3,2) we have;

$$9x^4 + 36y^4 + 4z^4 = 9w^4$$

$$(x, y, z, w) = (1,12,6,17)$$

<u>Case(C):</u>

$$pr^2X^4 + 4p(p+r)^2Y^4 + r^3Z^4 = r^2(p+r)W^4$$

$pr^2X^4 + 4p(p+r)^2Y^4 + r^3Z^4 = r^2(p+r)W^4$---(4)

pr^2*X^4 + 4p*(p+r)^2*Y^4 + r^3*Z^4 = r^2*(p+r)*W^4 --------(4)

p,r are arbitrary.

We use an identity p*(t+1)^4+q*(t)^4+r*(t^2+at+1)^2=(p+r)*(ct^2+dt+1)^2,

with (a,c,d,q) = $(-2p/r, (2p+r)/r, 0, 4p(r^2+p^2+2pr)/(r^2))$.

So, we look for the integer solutions $\{Z^2 = t^2 - \frac{2pt}{r} + 1,\ \ W^2 = (2p+r)t^2/r + 1\}$

By parameterizing the second equation and substituting the result to first equation, then we obtain quartic equation below.

$$u^2 = r^4k^4 + r(8pr^2 + 2r^3)k^2 + r(8pr^2 + 16p^{2r})k + r(4p^2r + 8p^3 + r^3)$$

This quartic equation is birationally equivalent to an elliptic curve below.

$$\begin{aligned} Y^2 + (16r^5p + 32r^4p^2)Y \\ &= X^3 + (8r^3p + 2r^4)X^2 + (-16r^6p^2 - 32r^5p^3 - 4r^8)X - 192r^9 * p^3 \\ &- 32r^{10}p^2 - 256r^8p\text{^}4 - 32r^{11}p - 8r^{12} \end{aligned}$$

The corresponding point is:

$$P(X, Y) = (\,-8pr^3 - 2r^4, -16r^5p - 32r^4p^2).$$

Hence we get:

$$2P(X, Y) = (\,6r^4 + 4pr^3 + p^2r^2, 16r^5p - 10r^4p^2 + 16r^6 - p^3r^3).$$

This point P is of infinite order, and the multiples mP, m = 2, 3, ...give infinitely many points. This quartic equation (4) has infinitely many parametric solutions below.

$$m = 2:$$

$$X = -8r^2 - 8pr + p^2$$

$$Y = 4(2r + p)r$$

$$Z = 3p^2 - 4pr - 8r^2$$

$$W = p^2 + 12pr + 8r^2$$

$$m = 3:$$

$$X = (p^2 + 12pr + 8r^2)(-8r^2 - 8pr + p^2)(-64r^4 - 128r^3p - 112p^2r^2 - 48p^3r + p^4)$$

$$Y = 4(2r + p)(3p^2 - 4pr - 8r^2)r(p^2 + 12pr + 8r^2)^2$$

$$Z = (p^2 + 12pr + 8r^2)(5p^6 - 84p^5r - 728r^2p^4 - 832r^3p^3 + 576r^4p^2 + 1280r^5p + 512r^6)$$

$$W = (p^2 + 12pr + 8r^2)(p^6 + 100p^5r + 104r^2p^4 + 576r^3p^3 + 1856r^4p^2 + 1792r^5p + 512r^6)$$

For (p,q)=(3,2) we have:

$$3x^4 + 75y^4 + 2z^4 = 5w^4$$

$$(x, y, z, w) = (71,56,26,113)$$

<u>Case(D):</u>

$$(-c^2 + 4ec - 2e^2)X^4 + (-c^2 + 4ec - 2e^2)Y^4 + (2c - 2e)^2 * Z^4 = 2W^4.$$

$$(-c^2 + 4ec - 2e^2)X^4 + (-c^2 + 4ec - 2e^2)Y^4 + (2c - 2e)^2 * Z^4 = 2W^4 .. (7)$$

We use an identity:

$$p(t + 1)^4 + p(t)^4 + r(t^2 + at + b)^2 = 2(ct^2 + dt + e)^2,$$

with $\{a, b, d, p, r\} =$

$$\{1, 1/2(c - 2e)(c - e), c, -c^2 + 4ec - 2e^2, (2c - 2e)^2\}.$$

So, we look for the integer solutions:

$$\{Z^2 = t^2 + t + \frac{1}{2}(c - 2e)(c - e), \quad W^2 = ct^2 + ct + e\}$$

By parameterizing the second equation and substituting the result to first equation, then we obtain quartic equation below.

$$u^2 = (-c^3 + 6c^2e - 8e^2c + 4e^3)k^4 + (8e^2c - 4c^2e)k^3 + (-28c^2e + 36e^2c + 6c^3 - 8e^3)k^2 + (8e^2c - 4c^2e)k - c^3 + 6c^2e - 8e^2c + 4e^3$$

This quartic equation has infinitely many rational solutions for |(p,r)|<100 below.

$$(c,e) = (7,4),(9,6),(9,7),(9,10),(9,13)$$

Hence we can obtain infinitely many integer solutions for equation (7) where

$$(c,e) = (7,4),(9,6),(9,7),(9,10),(9,13)$$

For (p,r)=(1,2) we get:

$$x^4 + 9y^4 + 2z^4 = 3w^4$$

$$(x,y,z,w) = (47,40,37,57)$$

Case (E):

$$a(p)^4 + b(q)^4 = c(r)^4 + d(s)^4$$

We take, (c=a) & (d=b) and we get:

$$a(p)^4 + b(q)^4 = a(r)^4 + b(s)^4 \quad --- (1)$$

Hence product of coeficents, $(abab) = (ab)^2$, equals a square

We multiply eqn. (1) by integer ten & we get

$$(10) * [a(p)^4 + b(q)^4] = (10) * [a(r)^4 + b(s)^4]$$

Hence we get:

$$(2)(5)(a)(p-r)(p+r)(p^2+r^2) = (2)(5)(b)(s-q)(s+q)(s^2+q^2)$$

Hence, we take, $(a,b) = \big((s-q),(p-r)\big)$

Let: $5(p+r) = 2(s+q)$

Applying the above conditions we get:

$$2(p^2 + r^2) = 5(s^2 + q^2) - - - (2)$$

Eqn.(1) has numerical solution at:

$$(p, q, r, s, a, b) = (11,2,-7,8,1,3) - - - - - - - - - - (3)$$

We parameterize eqn. (2) at:

$$[(11 + 2t), (2 + 5kt), (-7 + 2kt), (8 + 5t)]$$

& we get after removing common factors:

$$p = 33k^2 - 8k + 17$$

$$q = 14k^2 + 40k - 6$$

$$r = 29k^2 + 16k + 21$$

$$s = 24k^2 - 20k - 16$$

And Since,

$$(a, b) = \big((s - q), (p - r)\big)$$

we have,

$$a = 19k^2 + 10k - 11$$

$$b = 31k^2 + 4k + 19$$

For k=0 we get:

$$(p, q, r, s) = ((17,6,21,16) \ \ \& \ (a, b) = (-11,19)$$

$$11(17)^4 + 19(16)^4 = 11(21)^4 + 19(6)^4$$

In the above parameterization we used above eqn. (3) numerical solution. Similarly there are other numerical solutions listed below that can be used to arrive at more parametric forms for eqn. (1). This can be done by finding suitable relationship for the eqn.

$$m(p + r) = n(s + q)$$

Where, (m, n) are integers.

Numerical examples for equation (1):

$$(a, b, p, q, r, s) = (3,5,6,1,4,5)$$

$$(a,b,p,q,r,s) = (1,33,24,13,9,14)$$

$$(a,b,p,q,r,s) = (33,59,96,17,22,83)$$

$$(a,b,p,q,r,s) = (1,11,21,12,1,14)$$

$$(a,b,p,q,r,s) = (15,31,7,20,24,5)$$

$$(a,b,p,q,r,s) = (1,3,4,1,2,3)$$

$$(a,b,p,q,r,s) = (11,27,11,12,16,1)$$

$$(a,b,p,q,r,s) = (4,9,1,14,17,6)$$

– –

Case(F):

$$d^2X^4 + (d-2c)^2Y^4 = 2d(d-2c)Z^4 + 4W^4.$$

$$d^2X^4 + (d-2c)^2Y^4 = 2d(d-2c)Z^4 + 4W^4..(8)$$

We use an identity:

$p(t+1)^4 + q(t)^4 = r(t^2+at+b)^2 + s(ct^2+dt+e)^2,$

$$p = d^2, q = (d-2c)^2, r = 2d(d-2c), s = (2)^2$$

with $\{a,b,e,p,q,r\}$=

$$\{1,0,1/2(d),1/4(sd^2),1/4(sd^2) - scd + sc^2,$$

$$1/2(sd^2) - scd\}.$$

So, we look for the integer solutions:

$\left\{Z^2 = t^2 + t, \qquad W^2 = ct^2 + dt + \frac{1}{2}(d)\right\}$

By parameterizing the first equation and substituting the result to second equation, then we obtain quartic equation below.

$$u^2 = (4c-2d)k^4 + 2d$$

Quartic curve is transformed to the elliptic curve E.

$$x = (4c-2d)k^2, y = (4c-2d)uk.$$

$$E: y^2 = x^3 + 2(4c - 2d)dx$$

If rank of 'E' is greater than zero, then 'E' has infinitely many rational solutions.

Hence we can obtain infinitely many integer solutions for equation (8) if rank of E is greater than zero.

$$|c| < 5, d < 10$$

$$[c, d] rank\ [\,p, q, r, s] \quad [x, y, z, w]$$

We give below numerical solutions to the equation:

$$p(x)^4 + q(y)^4 = r(z)^4 + s(w)^4$$

$$[c, d] rank\ [\,p, q, r, s] \quad [x, y, z, w]$$

For (c,d)=(-4,1) & (p,q,r,s)=(1,81,18,4) we get:

$$x^4 + 81y^4 = 18z^4 + 4w^4$$

$$(x, y, z, w) = (9,1,3,6)$$

See Table below:

Table (1)

[-4, 1]	[1]	[1, 81, 18, 4]	[9, 1, 3, 6]
[-4, 1]	[1]	[1, 81, 18, 4]	[9801, 2209, 4653, 5106]
[-4, 5]	[2]	[25, 169, 130, 4]	[9, 1, 3, 14]
[-4, 5]	[2]	[25, 169, 130, 4]	[49, 9, 21, 74]
[-4, 5]	[2]	[25, 169, 130, 4]	[14161, 8649, 11067, 3886]
[-4, 8]	[1]	[16, 64, 64, 1]	[12769, 7056, 9492, 15934]
[-1, 2]	[1]	[1, 4, 4, 1]	[9, 4, 6, 7]
[-1, 2]	[1]	[1, 4, 4, 1]	[12769, 7056, 9492, 7967]
[-1, 8]	[1]	[16, 25, 40, 1]	[81, 16, 36, 158]

[0, 3]	[1]	[9, 9, 18, 4]	[49, 1, 7, 60]
[0, 7]	[1]	[49, 49, 98, 4]	[6241, 2209, 3713, 10920]
[1, 3]	[1]	[9, 1, 6, 4]	[9, 1, 3, 11]
[1, 3]	[1]	[9, 1, 6, 4]	[361, 625, 475, 13]
[1, 3]	[1]	[9, 1, 6, 4]	[201601, 27889, 74983, 246121]
[1, 4]	[1]	[4, 1, 4, 1]	[169, 1, 13, 239]

Table (Contd.):

$$p(x)^4 + q(y)^4 = r(z)^4 + s(w)^4$$

[c, d] rank [p, q, r, s]   [x,y,z,w]

Table (2):

[1, 8]	[1]	[16, 9, 24, 1]	[49, 16, 28, 94]
[1, 8]	[1]	[16, 9, 24, 1]	[3721, 1089, 2013, 7199]
[1, 9]	[2]	[81, 49, 126, 4]	[9, 1, 3, 19]
[1, 9]	[2]	[81, 49, 126, 4]	[25, 1, 5, 53]
[1, 9]	[2]	[81, 49, 126, 4]	[1521, 529, 897, 3071]
[1, 9]	[2]	[81, 49, 126, 4]	[3481, 2401, 2891, 5861]
[1, 9]	[2]	[81, 49, 126, 4]	[6241, 5329, 5767, 8711]
[1, 9]	[2]	[81, 49, 126, 4]	[42849, 47089, 44919, 22391]
[1, 9]	[2]	[81, 49, 126, 4]	[72361, 37249, 51917, 136771]
[4, 1]	[1]	[1, 49, -14, 4]	[2209, 1, 47, 1562]

[4, 2] [1]	[1, 9, -6, 1]	[1, 4, 2, 7]
[4, 2] [1]	[1, 9, -6, 1]	[121, 9, 33, 122]
[4, 2] [1]	[1, 9, -6, 1]	[169, 225625, 6175, 390794]
[4, 2] [1]	[1, 9, -6, 1]	[2209, 784, 1316, 2593]
[4, 3] [1]	[9, 25, -30, 4]	[1521, 529, 897, 2042]
[4, 5] [1]	[25, 9, -30, 4]	[529, 1521, 897, 2042]
[4, 6] [1]	[9, 1, -6, 1]	[4, 1, 2, 7]
[4, 6] [1]	[9, 1, -6, 1]	[9, 121, 33, 122]
[4, 6] [1]	[9, 1, -6, 1]	[225625, 169, 6175, 390794]

Note: Above Tables may facilitate further research in equation (1).

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